

# Periodic solutions of planetary systems with satellites and the averaging method in systems with slow and fast variables

The partial case of the planar  $N + 1$  body problem,  $N \geq 2$ , of the type of planetary system with satellites is studied. One of the bodies (the Sun) is assumed to be much heavier than the other bodies (“planets” and “satellites”), moreover the planets are much heavier than the satellites, and the “years” are much longer than the “months”. Under a nondegeneracy condition, which in general holds, the existence of at least  $2^{N-2}$  smooth 2-parameter families of symmetric periodic solutions in a rotating coordinate system is proved such that the distances between each planet and its satellites are much shorter than the distances between the Sun and the planets. Generating symmetric periodic solutions are described and necessity of the nondegeneracy condition is proved. Sufficient conditions for some periodic solutions to be orbitally stable in linear approximation are given. Via the averaging method, the results are extended to a class of Hamiltonian systems with slow and fast variables close to the systems of semidirect product type.

**Key words:**  $n$ -body problem, periodic solutions, orbital stability, averaging, slow and fast variables.

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In memory of Nikolai Nikolaevich Nekhoroshev

## § 1. Introduction

We study the partial case of the planar  $N + 1$  body problem,  $N \geq 2$ , that can be characterized as “the problem on the motion of a planetary system with satellites”.

An effective estimate for the number of smooth two-parameter families of symmetric periodic solutions of this problem in a rotating coordinate system is proved (theorems 1.1, 1.2(A) and corollary 1.1(Ξ) about “solutions of the first kind”). Sufficient conditions for orbital stability in linear approximation for some of these solutions are given (theorem 1.2(B)). Generating symmetric periodic solutions are described (theorem 1.1). The necessity of a nondegeneracy condition is proved (theorem 1.3

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and corollary 1.1( $\sharp$ )). The periodic solutions under our investigation are close to collections of independent “circular” solutions of the corresponding Kepler problems for each planet and each satellite. Via the averaging method (theorem 2.5 and corollary 2.1), the listed results are generalized to a wide class of Hamiltonian systems with slow and fast variables (theorems 2.1–2.4).

Theorems 1.1 and 1.2 of the present work include as partial cases results of G. Hill [1, 2, 3] and H. Poincaré [4] on the existence of periodic solutions and sufficient conditions of their orbital stability in linear approximation for the systems of the Sun–Earth–Moon and the Sun–two planets types (respectively). Theorem 1.1 implies the known results by G. A. Krassinsky [5] and E. A. Kudryavtseva [6, 7] on the number of periodic solutions of planetary systems without and with satellites (respectively), by V. N. Tkhai [8] on the number and the location of symmetric periodic solutions of the systems of the Sun–planets and Sun–planet–satellites types.

Let us formulate the results of the paper more precisely.

A general  $N + 1$  body problem is described by the system of equations

$$\mu_i \frac{d^2 \mathbf{r}_i}{dt^2} = -\frac{\partial U}{\partial \mathbf{r}_i}, \quad 0 \leq i \leq N, \quad (1)$$

where  $\mu_i$  is the mass of the  $i$ th body,  $\mathbf{r}_i$  is its radius vector in a fixed Euclidean space  $E$ ,

$$U = - \sum_{0 \leq i < j \leq N} \frac{g \mu_i \mu_j}{r_{ij}} \quad (2)$$

is the Newtonian potential of bodies’ attraction,  $r_{ij} = |\mathbf{r}_j - \mathbf{r}_i|$  ( $0 \leq i, j \leq N$ ) are pairwise distances between the bodies, and  $g > 0$  is the gravitational constant. We consider a *planar  $N + 1$  body problem*, i.e. the case of  $\dim E = 2$ .

REMARK 1.1. Without loss of generality, we may choose the unities of mass, distance and time as it will be suitable. In fact, for any constants  $a, b, c > 0$ , a collection of vector-functions  $\mathbf{r}_i(t)$  is a solution of the  $N + 1$  body problem with the gravitational constant  $g$  and the masses  $\mu_i$  if and only if the collection of vector-functions  $\tilde{\mathbf{r}}_i(\tilde{t}) := b^2 c \mathbf{r}_i(at\tilde{b}^3)$  is a solution of the  $N + 1$  body problem with the gravitational constant  $\tilde{g} = a^2 g$  and the masses  $\tilde{\mu}_i = c \mu_i$ . In particular, we do not need to assume that the gravitational constant  $g$  is arbitrary, but we may assume its value to be a distinguished number that we will choose below. (This can be achieved via scaling the time.)

DEFINITION 1.1. A solution of the planar  $N + 1$  body problem will be called *relatively periodic* (or simply *periodic*) if the locations of all bodies after the time-interval  $T > 0$  can be obtained from their initial locations by rotating the plane by the same angle  $\alpha$  around the centre of masses, for any initial time, where  $-\pi < \alpha \leq \pi$ . The pair of real numbers  $(T, \alpha)$  will be called *the relative period* of the solution, and the solution itself will be called  *$(T, \alpha)$ -periodic*.

Any solution obtained from a  $(T, \alpha)$ -periodic solution via shifting the time by a value  $t$  and rotating by an angle  $\varphi$  around the origin is a  $(T, \alpha)$ -periodic solution too. The union of the phase trajectories of all such solutions is a *two-dimensional torus* in the phase space, since it admits angular coordinates  $t \bmod T$ ,  $\varphi \bmod 2\pi$ . All

these solutions will be regarded as a single  $(T, \alpha)$ -periodic solution, and the union of their phase trajectories will be called *the phase orbit* of this solution.

Many relatively periodic solutions of the planar  $N + 1$  body problem happen to be *symmetric* (theorem 1.1). These solutions are characterized by the following property: at some time, all the bodies lie on the same line (i.e. a “parade” is observed) and their velocities are perpendicular to this line.

In the present work, the following partial case of the planar  $N + 1$  body problem is considered,  $N \geq 2$ . We assume that the mass of one particle (the Sun) equals  $\mu_0 = 1$  and is much greater than the masses of the other particles (the planets and satellites). Moreover the mass  $\mu_i$  of the  $i$ th planet and the mass  $\mu_{ij}$  of its  $j$ th satellite have the form

$$\mu_i = \mu m_i, \quad \mu_{ij} = \mu \nu m_{ij} \leq \mu_i, \quad 1 \leq i \leq n, \quad 1 \leq j \leq n_i, \quad (3)$$

where  $0 < \mu, \nu \ll 1$  are small parameters and  $m_i, m_{ij}$  are positive parameters far enough from zero (e.g. positive constants) with the properties

$$\sum_{i=1}^n m_i = 1, \quad \min_{i=1}^n \sum_{j=1}^{n_i} \frac{m_{ij}}{m_i} = 1, \quad (4)$$

moreover  $m_{ij}$  are bounded for  $n_i \geq 2$ , where  $n_i$  is the number of satellites of the  $i$ th planet and  $1 + n + \sum_{i=1}^n n_i = N + 1$  is the number of all bodies. Thus, for each “double planet” ( $n_i = 1$ ), the mass of the satellite equals  $\mu m_i \theta_i / (1 - \theta_i)$  where the parameter  $\theta_i := \nu m_{i1} / (m_i + \nu m_{i1}) \in (0, 1/2]$  is not necessarily small (since  $m_{i1}$  is not necessarily bounded). Due to (4), one has  $\theta_i(1 - \theta_i)/\nu = (1 - \theta_i)m_{i1}/\bar{m}_i \geq 1/4$  for  $n_i = 1$ .

We also assume that the distance  $R_i$  between the Sun and the  $i$ th planet is of order  $R \gg 1$ , while each satellite is at the distance  $r_{ij}$  of order 1 from its planet. Finally, “the years are much longer than the months”, i.e. the angular frequency  $\omega_i$  of the rotation of each planet around the Sun is of order  $\omega$ , while the angular frequencies  $\Omega_{ij}$  of the rotations of its satellites about it have order 1 where  $0 < \omega \ll 1$ . More precisely, let a set of non-vanishing real numbers

$$\omega_i = \omega \Omega_{i0}, \quad \Omega_{ij}, \quad 1 \leq i \leq n, \quad 1 \leq j \leq n_i, \quad (5)$$

called *the set of angular frequencies* satisfies the conditions

$$c \leq |\Omega_{i0}| \leq |\Omega_{ij}| \leq 1, \quad 1 \leq i \leq n, \quad 1 \leq j \leq n_i, \quad (6)$$

$$||\Omega_{i0}| - |\Omega_{i'0}|| \geq c, \quad i < i', \quad ||\Omega_{ij}| - |\Omega_{ij'}|| \geq c, \quad 1 \leq j < j' \leq n_i. \quad (7)$$

Here  $c$  is a suitable real number in the interval  $0 < c < 1$ .

Suppose that the parameters  $\omega, \mu, g, R$  satisfy the natural relations  $\omega^2 R^3 = g$  and  $1 = g\mu$  corresponding to Kepler’s second law for the planets ( $\omega_i^2 R_i^3 = g\mu_0$ ) and the satellites ( $\Omega_{ij}^2 r_{ij}^3 = g\mu_i$ ), for the chosen unities of mass, distance and time. Thus,  $g = \omega^2 R^3 = 1/\mu$ ,

$$\frac{1}{R^3} = \omega^2 \mu \quad (8)$$

and the problem has three independent small parameters:  $\mu, \nu$  and  $\omega$ . We emphasize that the initial  $N + 1$  body problem does not include the parameters  $\omega$  and  $R$ , so we

introduce them as additional small and big parameters. Therefore the only imposed restrictions to the parameters of the problem are as follows: the mass of the Sun is  $\mu_0 = 1$ , the distances from the satellites to their planets are of order 1, and the gravitational constant is  $g = 1/\mu$ . This does not cause any loss of generality due to remark 1.1.

Let us describe the “generating” periodic solutions of the  $N + 1$  body problem under consideration. These are the periodic solutions of the “model” problem (see remark 3.2 and the formula (73)) consisting of  $N$  independent Kepler’s problems, one for each planet or satellite. Let us assume that the collection (5) is maximally resonant, i.e. has the form

$$\begin{aligned}\omega_i &= \omega_1 + k_i \frac{2\pi}{T}, & 1 \leq i \leq n, \\ \Omega_{ij} &= \omega_1 + K_{ij} \frac{2\pi}{T}, & 1 \leq j \leq n_i,\end{aligned}\tag{9}$$

where  $k_i, K_{ij} \in \mathbb{Z}$ ,  $T > 0$ . The solutions of the model problem will be called *generating solutions* if they correspond to independent circular motions of the planets around the Sun (which is placed to the origin) and the satellites around the planets with the angular frequencies (5) of the form (6), (7), (9). Here the circular orbits of the planets and satellites have radii  $R_i = \frac{R}{|\Omega_{i0}|^{2/3}}$  and  $r_{ij} = \frac{m_i^{1/3}}{|\Omega_{ij}|^{2/3}}$  respectively,  $1 \leq i \leq n$ ,  $1 \leq j \leq n_i$ . The union of the phase trajectories of all such solutions is an  $N$ -dimensional torus, since the polar angles of  $N$  radius vectors drawn from the Sun to the planets and from the planets to their satellites can be used as coordinates on it. Denote this torus by  $\Lambda^\circ$ . Due to the condition (9), generating solutions are  $(T, \alpha)$ -periodic with relative period

$$T > 0, \quad \alpha = \omega_1 T + 2\pi k \in (-\pi, \pi]\tag{10}$$

where  $k \in \mathbb{Z}$  is a suitable integer. In the presence of satellites, the following condition will be called *the nondegeneracy condition*:

$$|\alpha| > \omega^2 T.\tag{11}$$

*Symmetric* periodic solutions of the three-body problem were studied already by Poincaré [4]. Recall the definition of a symmetric solution of the planar  $N + 1$  body problem.

**DEFINITION 1.2.** Consider a problem describing the motion of  $N + 1$  particles in a Euclidean plane. A solution of this problem will be called *symmetric* if there exists a line  $l$  in the plane, called the axis of symmetry, and a time  $t = t_0$  satisfying one of the following (equivalent) conditions called a “parade” of the particles:

- 1) at the time  $t = t_0$ , all points are placed on the line  $l$  (i.e. a “parade” of the particles is observed) and their velocities are orthogonal to the line  $l$ ;
- 2) the locations (and, hence, also the velocities) of all particles at any time  $t \in \mathbb{R}$  can be obtained from their locations at the time  $2t_0 - t$  by reflecting with respect to the axis  $l$ .

The particles of the system are assumed to be numbered. The order of them on the line  $l$  at the time  $t_0$  can be called *the type of a parade*. Any solution of the  $N + 1$  body problem obtained from a symmetric solution by shifting the time and

by rotating the plane is also symmetric. Similarly to the case of  $(T, \alpha)$ -periodic solutions, we will not distinguish such solutions and will regard them as a *single symmetric solution*.

One easily shows that exactly  $2^{N-2}$  of the generating solutions are symmetric. Namely, the symmetric  $(T, \alpha)$ -periodic solutions are characterized by the condition that “parades” of the planets and satellites are observed, moreover they repeat each half of the period,  $T/2$ . This means that all the particles of the system are posed on a line, which turns by the angle  $\alpha/2$  after the time-interval  $T/2$ .

**THEOREM 1.1.** *There exist constants  $\omega_0, C > 0$  and continuous positive functions  $\mu_0 = \mu_0(\omega, c)$  and  $\nu_0 = \nu_0(\omega, c)$  ( $0 < \omega \leq \omega_0$ ,  $0 < c < 1$ ) such that, for any parameter values  $\omega, c, \mu, \nu$  with the properties  $0 < \omega \leq \omega_0$ ,  $0 < c < 1$ ,  $0 < \mu \leq \mu_0(\omega, c)$  and  $0 < \nu \leq \nu_0(\omega, c)$  and any collection of angular frequencies (5) of the form (6), (7) and (9), the following property holds. Suppose that the parameters (10) satisfy either the nondegeneracy condition (11) or the following more delicate conditions:*

$$\alpha \neq 0, \quad \alpha - \frac{\omega_i^2}{4\Omega_{ij}}T \notin [-C\omega^3T, C\omega^3T] + 2\pi\mathbb{Z} \quad (12)$$

for  $1 \leq i \leq n$ ,  $1 \leq j \leq n_i$ . Then the  $N + 1$  body problem of the type of planetary system with satellites,  $N \geq 2$ , has exactly  $2^{N-2}$  symmetric  $(T, \alpha)$ -periodic solutions that are  $O(\omega^2)$ -close to the generating symmetric solutions corresponding to independent circular rotations of the planets and satellites with the angular frequencies (5). Each of these  $2^{N-2}$  solutions smoothly depends on the pair  $(T, \alpha)$ . For each of these solutions, parades are observed, which repeat each time-interval  $\frac{T}{2}$ : all of the particles of the system are posed on a line (which turns by the angle  $\alpha/2$  after the time-interval  $T/2$ ).

Let  $\Lambda^\circ \subset T^*Q$  be the  $N$ -dimensional torus in the phase space formed by the trajectories of the generating solutions (see (10)). Let  $\Sigma \subset T^*Q$  be a “transversal surface” of codimension 2 in the phase space, which transversally intersects invariant two-dimensional tori lying on  $\Lambda^\circ$  and corresponding to  $(T, \alpha)$ -periodic solutions:

$$\Sigma := \left\{ \sum_{i=1}^n (\varphi_i + \sum_{j=1}^{n_i} \varphi_{ij}) = \frac{T_{\min}}{T} \sum_{i=1}^n (k_i \varphi_i + \sum_{j=1}^{n_i} K_{ij} \varphi_{ij}) = 0 \pmod{2\pi} \right\}.$$

Here  $k_i, K_{ij}$  are integers in (9),  $T_{\min}$  is the minimal positive period, hence the integer  $T/T_{\min}$  is the greater common divisor of the collection of integers  $k_i, K_{ij}$ . Let  $\tilde{\Lambda}$  be the  $N$ -dimensional torus that is  $\omega^2$ -close to the torus  $\Lambda^\circ$  (see theorem 1.2 below). Let  $\Psi$  be the generating function of the “succession map”  $g_{H-\omega_1 I}^T$  of the  $N + 1$  body problem under consideration (see definition 2.1). Consider the smooth function  $\bar{S} = \Psi|_{\tilde{\Lambda} \cap \Sigma}$  on the  $(N - 2)$ -dimensional torus  $\tilde{\Lambda} \cap \Sigma$ . Since the function  $\bar{S}$  is defined on a  $(N - 2)$ -dimensional torus, it has at least  $N - 1$  critical points, moreover at least  $2^{N-2}$  points counted with multiplicities [9]. We will prove (see theorem 1.2) the same lower bound for the number of  $(T, \alpha)$ -periodic solutions of the problem under consideration. Observe that the function  $\bar{S}$  has at least one critical point, since it is defined on a closed manifold. We will prove that each critical point of

the function  $\bar{S}$  corresponds to a  $(T, \alpha)$ -periodic solution of the  $N + 1$  body problem. Moreover, we will offer sufficient conditions that guarantee the *orbital (structural) stability in linear approximation* (see definition 3.1) of such a solution.

The following condition will be called the *strong nondegeneracy condition*:

$$\omega^2 T < |\alpha| < \pi - \omega^2 T. \quad (13)$$

The following conditions will be called the *property of having fixed sign*:

1) all planets rotate “to the same side”, i.e. the angular frequencies of the rotations of the planets around the Sun have the same sign:

$$\omega_i \omega_{i'} > 0, \quad 1 \leq i < i' \leq n; \quad (14)$$

2) the function  $\bar{S}$  on the  $(N - 2)$ -dimensional torus is either a Morse function or has at least one nondegenerate critical point of a local minimum (this condition is assumed to be always true if  $N = 2$ ).

**THEOREM 1.2.** *Under the hypothesis of theorem 1.1, there exists a smooth  $N$ -dimensional torus  $\tilde{\Lambda}$  in the phase space that is  $\omega^2$ -close to the torus  $\Lambda^\circ$ , smoothly depends on the pair  $(T, \alpha)$  and has the following properties.*

(A) *The phase orbits of all  $(T, \alpha)$ -periodic solutions of the  $N + 1$  body problem that are  $\omega$ -close to the torus  $\Lambda^\circ$  are contained in the torus  $\tilde{\Lambda}$ . Moreover their intersection points with the transversal surface  $\Sigma$  coincide with critical points of the function  $\bar{S} := \Psi|_{\tilde{\Lambda} \cap \Sigma}$  on the  $(N - 2)$ -dimensional torus  $\tilde{\Lambda} \cap \Sigma$ . Here  $\Psi$  is the generating function of the “succession map”  $g_{H-\omega_1 I}^T$  of the problem under consideration (see definition 2.1). The function  $\Psi|_{\tilde{\Lambda}}$  is an even function in the collection  $\varphi$  of angle variables  $\varphi_i|_{\tilde{\Lambda}}$ ,  $\varphi_{ij}|_{\tilde{\Lambda}}$ . The phase orbits of the symmetric  $(T, \alpha)$ -periodic solutions contain the points  $\varphi$  of the torus  $\tilde{\Lambda}$  having the property  $\varphi = -\varphi$ .*

(B) *Suppose that the property of having fixed sign holds, moreover either the strong nondegeneracy condition (13) or the following more delicate conditions hold:*

$$\alpha \notin \{0, \pi\}, \quad \frac{\operatorname{sgn} \omega_1 + \operatorname{sgn} \Omega_{ij}}{2} \alpha - \frac{\omega_i^2}{8|\Omega_{ij}|} T \notin \left[ -\frac{C}{2} \omega^3 T, \frac{C}{2} \omega^3 T \right] + \pi \mathbb{Z}, \quad (15)$$

$$\frac{\operatorname{sgn} \Omega_{ij} + \operatorname{sgn} \Omega_{i'j'}}{2} \alpha - \left( \frac{\omega_i^2}{|\Omega_{ij}|} + \frac{\omega_{i'}^2}{|\Omega_{i'j'}|} \right) \frac{T}{8} \notin [-C \omega^3 T, C \omega^3 T] + \pi \mathbb{Z} \quad (16)$$

for all  $1 \leq i, i' \leq n$ ,  $1 \leq j \leq n_i$  and  $1 \leq j' \leq n_{i'}$ . Then the  $(T, \alpha)$ -periodic solution corresponding to any nondegenerate critical point of a local minimum of the function  $\bar{S}$  is orbitally structurally stable in linear approximation.

Theorem 1.2(B) implies that, for  $N = 3$ , in the “generic case”, a half of the  $(T, \alpha)$ -periodic solutions that are close to the torus  $\Lambda^\circ$  are orbitally stable in linear approximation (since the function  $\bar{S}$  is defined on a circle and, hence, has only critical points of local minima and maxima, which alternate on the circle).

The inequality  $|\alpha| \leq \pi$  and the nondegeneracy condition (11) (respectively (12)) imply that, in the presence of satellites, the period  $T$  of the solutions under consideration is “not too big”: respectively

$$T < \frac{\pi}{\omega^2} \quad \text{or} \quad T < \frac{\pi}{C \omega^3}. \quad (17)$$

REMARK 1.2. (A) One easily checks that the nondegeneracy condition (11) is always realizable (for suitable collections of the angular frequencies  $\omega_i, \Omega_{ij}$ ). In fact, for any  $0 < \omega \leq \omega_0(a) := \frac{1}{4a}$ , the period  $T$  can take any value of the form

$$T = \frac{2\pi a}{\omega}, \quad a \geq a_0(k_2, \dots, k_n) := \max \left\{ 7|k_i|, \sqrt{7(N-n+1)} \right\}, \quad (18)$$

thus the value  $\omega^2 T$  can be arbitrary small, hence any of the nondegeneracy conditions (11) and (12) can be always achieved. In more detail, let us fix any integers  $k_2, \dots, k_n \in \mathbb{Z} \setminus \{0\}$  having different absolute values, as well as any values  $a \geq a_0(k_2, \dots, k_n)$  and  $c \in (0, c_0(a)]$  where  $c_0(a) := \min\{\frac{1}{a}, \frac{1}{14(N-n+1)}\}$ . For any  $b \in [\frac{2}{7}, \frac{3}{7}]$  and  $\omega \in (0, \omega_0(a)]$  put

$$\omega_1 := \omega b, \quad \omega_i := \omega_1 + k_i \frac{\omega}{a}, \quad \Omega_{ij} := \omega_1 + K_{ij} \frac{\omega}{a} \quad (19)$$

for any integers  $K_{ij} \in \mathbb{Z}$  with the properties  $|K_{ij}| \in [\frac{5a}{7\omega}, \frac{6a}{7\omega}]$  and  $||K_{ij}| - |K_{ij'}|| \geq \ell$  (for all  $i, j \neq j'$ ) where  $\ell \in [\frac{ca}{\omega}, \frac{a}{7(N-n+1)\omega}] \cap \mathbb{N}$ . Such integers  $\ell$  and  $K_{ij}$  exist for  $0 < \omega \leq \omega_0(a) = \frac{1}{4a}$ , since the interval  $[\frac{ca}{\omega}, \frac{a}{7(N-n+1)\omega}]$  (respectively  $[\frac{5a}{7\omega}, \frac{6a}{7\omega}]$ ) is of the length  $\frac{a}{\omega}(\frac{1}{7(N-n+1)} - c) \geq \frac{2a^2}{7(N-n+1)} \geq 2$  (respectively  $\frac{a}{7\omega} \geq (N-n+1)\ell$ ) and, hence, it always contains a positive integer (respectively  $N-n$  different positive integers with pairwise distances  $\geq \ell$ ). Since  $\alpha = \omega_1 T = 2\pi ab \pmod{2\pi}$  and  $\omega^2 T = 2\pi a\omega \leq \frac{\pi}{2}$ , the required nondegeneracy condition (11) has the form  $(a(b-\omega), a(b+\omega)) \cap \mathbb{Z} = \emptyset$ . Hence it holds if  $(ab - \frac{1}{4}, ab + \frac{1}{4}) \cap \mathbb{Z} = \emptyset$ , i.e.  $b \in \frac{1}{2a} + [-\frac{1}{4a}, \frac{1}{4a}] + \frac{1}{a}\mathbb{Z}$ . It follows from  $\frac{1}{7} \geq \frac{1}{2a}$  that, for any  $a \geq a_0(k_2, \dots, k_n)$ , there exists  $b \in [\frac{2}{7}, \frac{3}{7}] \cap ([\frac{1}{4a}, \frac{3}{4a}] + \frac{1}{a}\mathbb{Z})$ . Clearly the nondegeneracy condition (11) holds for any such  $b$ . The obtained real numbers  $\omega_i, \Omega_{ij}$  satisfy the inequalities  $c < \frac{1}{7} \leq \frac{|\omega_i|}{\omega} \leq \frac{4}{7} < |\Omega_{ij}| < 1$ ,  $\frac{||\omega_i| - |\omega_{i'}||}{\omega} \geq \frac{1}{a} \geq c$ ,  $||\Omega_{ij}| - |\Omega_{ij'}|| \geq \frac{\omega}{a}\ell \geq c$  and, hence, the inequalities (6) and (7). This proves the realizability of any period  $T$  of the form (18). In particular, for any

$$a \geq a_0(k_2, \dots, k_n), \quad b = b(a) := \frac{1}{a} \left[ \frac{6a-7}{14} \right] + \frac{1}{2a}, \quad (20)$$

the condition  $b \in [\frac{2}{7}, \frac{3}{7}]$  and the nondegeneracy condition (11) automatically hold for the period (18) and the collection of angular frequencies (19). Thus the system of relations (18), (19) and (20) (called *the rough nondegeneracy condition*) implies the nondegeneracy. Due to (6), the nondegeneracy (11) implies the delicate nondegeneracy (12) for small enough  $0 < \omega \ll 1$ .

(B) In the case  $n > 1$  (of systems with *more than one planet*), the relative period  $T$  is always “big”, namely  $T = \frac{2\pi|k_2|}{|\omega_2 - \omega_1|} > \frac{\pi|k_2|}{\omega} \geq \frac{\pi}{\omega} \gg 1$  for  $0 < \omega \ll 1$ , due to (9), (6) and (7). Together with any of the inequalities (17) (which are corollaries of the nondegeneracy condition (11) or (12)), this implies that the period  $T$  under consideration belongs to the interval  $(\frac{\pi}{\omega}, \frac{\pi}{\omega^2})$  or  $(\frac{\pi}{\omega}, \frac{\pi}{C\omega^3})$  respectively. In the case  $N = 2, n = 1$  (of a system of the Sun–Earth–Moon type), the minimal positive relative period is bounded and equals  $T_{\min} = 2\pi/|\Omega_{11} - \omega_1| \leq 4\pi/c$ , while the rotation angle  $\alpha_{\min} = \omega_1 T_{\min}$  is of order  $\omega$ . Hence the nondegeneracy condition (11) automatically holds for  $T = T_{\min}$ . In the case  $n = 1, N > 2$  (of a system of the



Sun–planet–satellites type), the minimal positive relative period can be bounded too. However we do not assume in theorems 1.1–1.3 that the period  $T > 0$  in (10) is minimal. We only assume that it satisfies either the nondegeneracy condition (11) or the delicate condition (12) or the rough condition (18), (19), (20). In particular, we can assume that  $T = \frac{\text{const}}{\omega}$ , see (A).

The natural question arises: is the nondegeneracy condition (11) necessary for the validity of theorem 1.1? An answer happens to be affirmative in many cases.

In the following theorem, by “almost any” collection of masses  $\mu_i > 0$ ,  $1 \leq i \leq n$ , we mean any collection belonging to the complement in  $\mathbb{R}_{>0}^n$  to the union of a finite set of linear subspaces of  $\mathbb{R}^n$ . Moreover each of these subspaces depends on the collection of integers  $k_i$  in (9), has codimension at least 2, and the number of these subspaces does not exceed  $2^{n-2}$ . The set  $\mathcal{M}^{\text{sym}}$  of “almost all” collections of masses is described in more detail in §4.1. By the phase space of satellites, we regard the direct product of big balls in the phase spaces of the corresponding Kepler problems, except for a small neighbourhood of “the set of possible collisions”.

**THEOREM 1.3.** *Suppose that, under the hypothesis of theorem 1.1, the number  $n$  of planets is at least 2 and there exist two planets whose angular frequencies satisfy the following resonance relation:*

$$\frac{\omega_i}{\omega_{i'}} \in \left\{ \frac{k}{k+1} \mid k \in \mathbb{Z} \setminus \{-1, 0\} \right\}. \quad (21)$$

*In this case,  $\alpha = 0$  automatically. Then there exist an open dense subset  $\mathcal{M}^{\text{sym}} \subset \mathbb{R}_{>0}^n$  and a nonempty open subset  $\mathcal{M} \subset \mathcal{M}^{\text{sym}}$  (see definition 4.1 and remark 4.1), both invariant under multiplication by any positive real number and having the following properties. For “almost any” collection of planets’ masses  $\mu m_i > 0$ , namely for any collection of planets’ masses  $\mu(m_1, \dots, m_n) \in \mathcal{M}^{\text{sym}}$  (respectively for any collection of planets’ masses  $\mu(m_1, \dots, m_n) \in \mathcal{M}$ , for example satisfying the inequality  $|\kappa_{ii'}|c_{\kappa_{ii'}}m_{i'} > \sum_{l \neq i, i'} |\kappa_{il}|c_{\kappa_{il}}m_l$ , see (81), (82), (83)), there exist numbers  $\mu_0, \nu_0 > 0$  and an open subset  $U_0$  in the phase space of the planets containing the phase orbits of all symmetric “circular” solutions (respectively all circular solutions) of the collection of the Kepler problems for planets with angular frequencies (5), such that the following condition holds. For any values  $\mu, \nu, \tilde{T}, \tilde{\alpha} \in \mathbb{R}$  of the form*

$$0 < \left( \frac{\nu}{\nu_0} \right)^3 \leq \mu \leq \mu_0, \quad |\tilde{T} - T| + |\tilde{\alpha}| \leq D\mu,$$

*there exists no  $(\tilde{T}, \tilde{\alpha})$ -periodic solution of the  $N+1$  body problem under consideration whose phase orbit has a nonempty intersection with the direct product  $U$  of  $U_0$  and the phase space of the satellites.*

In particular, the region  $U$  does not contain the phase orbit of any *symmetric*  $T$ -periodic solution (respectively  $T$ -periodic solution).

Consider the planetary system with two planets, a partial case of the three-body problem. In this case, the minimal positive period  $T_{\min}$  equals  $\frac{2\pi}{|\omega_2 - \omega_1|}$ , thus the condition (21) means that the corresponding angle  $\alpha_{\min} = \frac{2\pi\omega_1}{\omega_2 - \omega_1} - 2\pi k$  vanishes. We also observe that, in this case, the region  $U$  in theorem 1.3 contains the whole



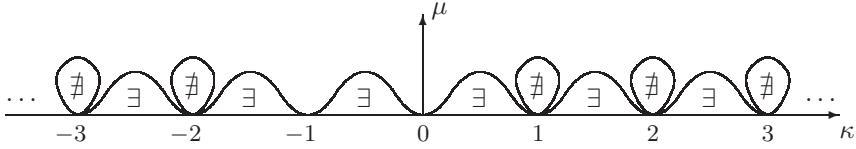


Fig. 1. A set of pairs  $(\kappa = \frac{\omega_1}{\omega_2 - \omega_1}, \mu)$  such that there exists ( $\exists$ ) or does not exist ( $\nexists$ ) a relatively-periodic solution of the 3-body problem

two-dimensional torus  $\Lambda^\circ$  and  $\mathcal{M}^{\text{sym}} = \mathcal{M} = \mathbb{R}_{>0}^2$  (i.e. “almost any” means “any”). Thus, theorems 1.1 and 1.3 for  $N = n = 2$  imply the following.

**COROLLARY 1.1.** *Consider the three-body problem of the type of planetary system with two planets. Fix angular frequencies of planets  $\omega_1 \neq 0$ ,  $\omega_2 \neq 0$ ,  $|\omega_1| \neq |\omega_2|$ , and consider the two-dimensional torus  $\Lambda^\circ$  corresponding to the circular motions of planets with frequencies  $\omega_1, \omega_2$ . Put  $T := \frac{2\pi}{|\omega_2 - \omega_1|}$ ,  $\alpha = 2\pi \frac{\omega_1}{\omega_2 - \omega_1} + 2\pi k \in (-\pi, \pi]$  for a suitable  $k \in \mathbb{Z}$ . In dependence on the ratio of these frequencies, one of the following statements holds.*

( $\exists$ ) *Suppose that the angular frequencies  $\omega_1, \omega_2$  does not satisfy the special resonance condition (21). Then  $\alpha \neq 0$  and there exists a number  $\mu_0 = \mu_0(m_1, m_2, \omega_1, \omega_2) > 0$  such that, for any values  $\mu, \tilde{\omega}_1, \tilde{\omega}_2$ ,  $|\mu| + |\tilde{\omega}_1 - \omega_1| + |\tilde{\omega}_2 - \omega_2| \leq \mu_0$ , there exists a two-dimensional torus  $\Lambda_{\mu, \tilde{\omega}_1, \tilde{\omega}_2}$  that smoothly depends on the triple  $(\mu, \tilde{\omega}_1, \tilde{\omega}_2)$ , coincides with the torus  $\Lambda^\circ$  if  $(\mu, \tilde{\omega}_1, \tilde{\omega}_2) = (0, \omega_1, \omega_2)$  and has the following property. If  $0 < \mu \leq \mu_0$  then the torus  $\Lambda_{\mu, \tilde{\omega}_1, \tilde{\omega}_2}$  is the phase orbit of a symmetric  $(\tilde{T}, \tilde{\alpha})$ -periodic solution of the problem under consideration with parameters  $\tilde{T} = \frac{2\pi}{|\tilde{\omega}_2 - \tilde{\omega}_1|}$ ,  $\tilde{\alpha} = 2\pi \frac{\tilde{\omega}_1}{\tilde{\omega}_2 - \tilde{\omega}_1} + 2\pi \tilde{k}$ , for a suitable  $\tilde{k} \in \mathbb{Z}$ .*

( $\nexists$ ) *Suppose that the angular frequencies  $\omega_1, \omega_2$  are in a special resonance (21). Then  $\alpha = 0$  and, for any numbers  $T, D > 0$  with  $\frac{\omega_2 - \omega_1}{2\pi} T \in \mathbb{Z}$ , there exist a number  $\mu_0 = \mu_0(m_1, m_2, \omega_1, \omega_2, T, D) > 0$  and a neighbourhood  $U = U_{m_1, m_2, \omega_1, \omega_2, T, D}$  of the torus  $\Lambda^\circ$  in the phase space such that, for any parameter value  $\mu \in (0, \mu_0]$  of the three-body problem under consideration,  $U$  does not contain any  $(\tilde{T}, \tilde{\alpha})$ -periodic orbit with parameters  $\tilde{T}, \tilde{\alpha}$  of the form*

$$|\tilde{T} - T| + |\tilde{\alpha}| \leq D\mu.$$

The figure shows (for a fixed  $\omega_1 \neq 0$ ) regions in the plane  $\mathbb{R}^2$  consisting of pairs  $(\kappa, \mu) \in \mathbb{R} \times \mathbb{R}_{>0}$ ,  $\kappa := \frac{\omega_1}{\omega_2 - \omega_1}$ ,  $0 < \mu \leq \mu_0(m_1, m_2, \omega_1, \omega_2)$ , such that there exists (respectively does not exist) a relatively-periodic solution of the three-body problem with parameters  $T = \frac{2\pi}{|\omega_2 - \omega_1|}$ ,  $\alpha = 2\pi\kappa + 2\pi k$ , for a suitable  $k \in \mathbb{Z}$ .

Due to corollary 1.1, for the planetary system with two planets, the condition (21) is false if and only if there exists a  $(T, \alpha)$ -periodic solution close to a “circular”  $(T, \alpha)$ -periodic motion. These  $(T, \alpha)$ -periodic solutions were discovered already by H. Poincaré [4] who called them *solutions of the first kind*. In the degenerate case (21), Poincaré discovered periodic solutions corresponding to elliptic motions of the planets, *solutions of the second kind*.

Theorems 1.1–1.3 seem to admit an extension to the cases of the  $N + 1$  body problem on a sphere or on the Lobachevskiy plane, due to periodicity of solutions of the Kepler problems on these surfaces (see [10] and references therein).

The paper is organized as follows. In §2, theorems 2.1–2.4 on periodic solutions of Hamiltonian systems with slow and fast variables are formulated, and a scheme of their proof is sketched via generalizing the averaging method (see theorem 2.5 and corollary 2.1) and the method of generating function (see definition 2.1). In §3.1 and §3.2, the parameters and the phase space of the  $N + 1$  body problem under consideration are described in more detail and the notions of a symmetric solution and an orbitally stable in linear approximation solution are discussed (see definition 3.1). In §3.3 and §3.4, we introduce coordinates in the phase space of the  $N + 1$  body problem, which bring the system to the form of the systems studied in §2 (lemmata 3.1 and 3.3), moreover we establish relations between our unperturbed system, the Hill problem, the three-body problem and the restricted three-body problem (§3.3.2). In §4, theorems 1.1–1.3 are derived from theorems 2.1–2.4.

A proof of the technical theorem 2.5, its corollary 2.1 and deriving from them theorems 2.1–2.4 will be published in our future work.

As a conclusion, we remark that the question of interpreting the discovered class of relatively-periodic solutions of the  $N + 1$  body problem in terms of the behaviour of planets and satellites of the real solar system is very interesting, needs an additional investigation, and it is not discussed in the present paper.

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## § 2. The averaging method for a class of systems with slow and fast variables

In §3 and §4, we will derive theorems 1.1–1.3 from the next theorems 2.1–2.4 on periodic solutions of dynamical systems having the following special form.

Let  $p : M \rightarrow M_0$  be a locally trivial fibre bundle of smooth manifolds. Fix a number  $\lambda \in \mathbb{R}$ . A pair of differential  $k$ -forms  $(\omega, \omega_1)$  on  $M$  will be called a  $\lambda$ -pair if the  $k$ -form  $\omega_0 := \omega - \lambda\omega_1$  “projects” to the base  $M_0$  of this fibre bundle (i.e. has the form  $\omega_0 = p^*\hat{\omega}_0$  for some  $k$ -form  $\hat{\omega}_0$  on  $M_0$ ),  $k \geq 0$ . A  $\lambda$ -pair  $(\omega, \omega_1)$  of 2-forms on  $M$  will be called a  $\lambda$ -symplectic structure if the 2-form  $\hat{\omega}_0$  is a symplectic structure on  $M_0$ ,  $\omega_1$  is closed and

$$T_x M = (\text{Ker } \omega_0|_x) \oplus (\text{Ker } \omega_1|_x)$$

at any point  $x \in M$ . (Thus the 2-form  $\omega_1$  determines a symplectic structure on each fibre, moreover its field of kernels is transversal to the fibre at any point of  $M$  and determines a “symplectic” flat connection of the fibre bundle  $p$ ). In this case, the fibre bundle  $p$  will be called a *symplectic semidirect product* and denoted by  $(M, M_0, p; \omega_0, \omega_1)$ . A vector field  $v = v_{H, H_1}$  on  $M$  will be called  $\lambda$ -Hamiltonian if

$$(\omega_0(\cdot, v) - dH)|_{\text{Ker } \omega_1} = 0, \quad (\omega_1(\cdot, v) - dH_1)|_{\text{Ker } \omega_0} = 0$$

for some  $\lambda$ -symplectic structure  $(\omega, \omega_1)$  and  $\lambda$ -pair of functions  $(H, H_1)$  on  $M$ . In this case, the dynamical system  $\dot{x}(t) = v(x(t))$  on  $M$  will be called  $\lambda$ -Hamiltonian

and denoted by

$$(M, M_0, p; \omega_0, \omega_1; H, H_1)^\lambda, \quad (22)$$

moreover the functions  $H$  and  $H_1$  will be called *the Hamiltonian function* and *the  $\lambda$ -Hamiltonian function* (respectively) of the system (22). If  $\lambda \neq 0$  then the system (22) is equivalent to the Hamiltonian system  $(M, \omega, H)$ :

$$(M, M_0, p; \omega_0, \omega_1; H, H_1)^\lambda \cong (M, \omega, H).$$

Here the symbol  $\cong$  denotes the equivalence of  $(\lambda)$ -Hamiltonian systems, i.e. the coincidence of the corresponding  $(\lambda)$ -Hamiltonian vector fields. If  $\lambda = 0$  then the system (22) is a *semidirect product*. (Such systems are studied by Yu. M. Vorobiev [11]; they include e.g. the restricted three-body problem, see §3.3.2).

Denote by  $g_{H, H_1}^t$  the flow of the  $\lambda$ -Hamiltonian vector field  $v_{H, H_1}$ . Similarly to the case of Hamiltonian systems, the flow of the field  $v_{H, H_1}$  always preserves the 2-form  $\omega$  and the Hamiltonian function  $H$ .

As a very special example, consider the case when  $M = M_0 \times M_1$  is a direct product, moreover the 2-form  $\omega_1$  and the function  $H_1$  “project” to  $M_1$  (i.e. have the form  $\omega_1 = p_1^* \hat{\omega}_1$  and  $H_1 = \hat{H}_1 \circ p_1$  for some 2-form  $\hat{\omega}_1$  on  $M_1$  and function  $\hat{H}_1 \in C^\infty(M_1)$  where  $p_1 : M \rightarrow M_1$  is the projection). Then the  $\lambda$ -Hamiltonian system (22) for any  $\lambda \in \mathbb{R}$  is equivalent to the Hamiltonian system  $(M, \omega_0 + \omega_1, H_0 + H_1)$ , i.e. to the *direct product* of the Hamiltonian systems  $(M_0, \hat{\omega}_0, \hat{H}_0)$  and  $(M_1, \hat{\omega}_1, \hat{H}_1)$ :

$$(M, M_0, p; \omega_0, \omega_1; H, H_1)^\lambda \cong (M, \omega_0 + \omega_1, H_0 + H_1). \quad (23)$$

Let us describe a class of dynamical systems *with slow and fast variables*. Consider a two-parameter family of  $\varepsilon$ -Hamiltonian systems on  $M$  with  $\varepsilon$ -symplectic structure  $(\omega, \omega_1)$ , the Hamiltonian  $H = \omega H_0 + \varepsilon H_1$  and the  $\varepsilon$ -Hamiltonian  $H_1$  where  $|\varepsilon|, |\omega| \ll 1$ . Here the 2-forms  $\omega_0, \omega_1$  and the functions  $H_0 = \hat{H}_0 \circ p$ ,  $H_1$  depend, in general, on the small parameters  $\varepsilon, \omega$  and possibly on some other parameters of the system, moreover some relations between parameters may be posed. The local coordinates of a point  $x_0 \in M_0$  are “slow variables”, while the local coordinates of a point “on a fibre” are “fast variables” of the system.

EXAMPLE 2.1. The following problems are described by systems with slow and fast variables:

- (i) the problem on the motion of a charged particle in a strong magnetic field on a symplectic manifold  $(M_0, \omega_0)$  where  $M = T^*M_0$  is the cotangent bundle, the magnetic field is given by the 2-form  $\varepsilon^{-1/4} \hat{\omega}_0$  on  $M_0$ , and the small parameters  $\omega, \varepsilon$  are related by the condition  $\omega^2 = \varepsilon$ ;
- (ii) the  $N + 1$  body problem of the type of “planetary system with satellites” where  $M = M_0 \times M_1$  is the direct product of the phase spaces of planets and satellites,  $\omega_1 = p_1^* \hat{\omega}_1$ ,  $\omega = \omega_0 + \varepsilon \omega_1$ , and the Hamiltonian function is

$$\tilde{H} = \omega \tilde{H}_0(x_0; \mu) + \varepsilon \left( \tilde{H}_1(x_1; \nu) + \omega^2 \tilde{\Phi}(x_0, x_1; \mu, \nu, \rho) \right).$$

Here  $(x_0, x_1) \in M_0 \times M_1$ ,  $0 < \omega, \varepsilon, \mu, \nu, \rho \ll 1$  are small parameters related by the conditions  $\rho = \omega^{2/3} \mu^{1/3}$  and  $\varepsilon = \omega^{1/3} \mu^{2/3} \nu$ . The planar  $N + 1$  body problem is  $S^1$ -symmetric and reversible (see below).

From now on, we assume that  $M = M_0 \times M_1$  is the direct product and  $\omega_1 = p_1^* \hat{\omega}_1$ .

Suppose that each symplectic manifold  $(M_i, \hat{\omega}_i)$  is equipped with the Hamiltonian action of a circle  $SO(2) = S^1 = \mathbb{R}/2\pi\mathbb{Z}$  with the Hamiltonian function  $I_i$ ,  $i = 0, 1$ . The system (22) will be called  $S^1$ -symmetric (or  $SO(2)$ -symmetric) if the functions  $H, H_1$  are invariant under the diagonal action of the circle on  $M$  (i.e. invariant under the flow of the  $\lambda$ -Hamiltonian field  $v_{I, I_1}$  on  $M$  where  $I = I_0 + \lambda I_1$ ). All solutions of this system that differ by shifts along the commuting vector fields  $v_{H, H_1}$  and  $v_{I, I_1}$  will be regarded as a single solution, and the union of their phase trajectories will be called *the phase orbit* of this solution. Let  $T, \alpha \in \mathbb{R}$ ,  $T \neq 0$ . A solution  $\gamma(t)$  of a  $S^1$ -symmetric system will be called  $(T, \alpha)$ -periodic if it is defined on the whole time-axis, and  $\gamma(t) = g_{H, H_1}^T g_{I, I_1}^{-\alpha}(\gamma(t))$  for some (and, hence, for any)  $t \in \mathbb{R}$ .

A  $S^1$ -symmetric system (22) will be called *reversible* (or  $O(2)$ -symmetric) if each  $M_i$  is equipped with an anti-canonical involution  $J_i : M_i \rightarrow M_i$  preserving the function  $I_i$  (i.e.  $J_i^* \hat{\omega}_i = -\hat{\omega}_i$  and  $I_i \circ J_i = I_i$ ),  $i = 0, 1$ , moreover the functions  $H, H_1$  are invariant under the (component-wise anti-canonical) involution  $J := J_0 \times J_1 : M \rightarrow M$ . A solution  $\gamma(t)$  of the reversible system will be called *symmetric* if it is defined on a time-interval  $(t_0 - \varepsilon, t_0 + \varepsilon) \subset \mathbb{R}$  and  $\gamma(t_0) = g_{I, I_1}^{\varphi_0} J g_{I, I_1}^{-\varphi_0}(\gamma(t_0))$  for some  $\varphi_0 \in \mathbb{R} \bmod 2\pi$  (and, hence,  $\gamma(2t_0 - t) = g_{I, I_1}^{\varphi_0} J g_{I, I_1}^{-\varphi_0}(\gamma(t))$  for any  $t \in (t_0 - \varepsilon, t_0 + \varepsilon)$ ).

Let us describe the *model system* on  $M = M_0 \times M_1$ :

$$(M, M_0, p; \omega_0, \omega_1; \omega H_0, H_1)^0 \cong (M, \omega_0 + \omega_1, \omega H_0 + H_1), \quad (24)$$

$0 < \omega \ll 1$  where  $H_1 = \hat{H}_1 \circ p_1$  for some function  $\hat{H}_1 \in C^\infty(M_1)$ , see (22) and (23). We will assume that each of the Hamiltonian systems  $(M_0, \hat{\omega}_0, \hat{H}_0)$  and  $(M_1, \hat{\omega}_1, \hat{H}_1)$  in the system (24) is the direct product of  $S^1$ -symmetric Hamiltonian systems:

$$(M_0, \hat{\omega}_0, \hat{H}_0) = \prod_{i=1}^n (M_{i0}, \hat{\omega}_{i0}, H_{i0}), \quad (M_1, \hat{\omega}_1, \hat{H}_1) = \prod_{i=1}^n \prod_{j=1}^{n_i} (M_{ij}, \hat{\omega}_{ij}, H_{ij}).$$

Moreover each factor  $M_{ij} = S^1 \times (a_{ij}, b_{ij}) \times \mathbb{R}^2$  is equipped with coordinates  $\varphi_{ij} \bmod 2\pi, I_{ij}, q_{ij}, p_{ij}$  such that  $\hat{\omega}_{ij} = dI_{ij} \wedge d\varphi_{ij} + dp_{ij} \wedge dq_{ij}$ , the action of the circle on  $(M_{ij}, \hat{\omega}_{ij})$  is given by the Hamiltonian  $I_{ij}$ , and the involution  $J$  acts component-wise in the form  $(\varphi_{ij}, I_{ij}, q_{ij}, p_{ij}) \mapsto (-\varphi_{ij}, I_{ij}, q_{ij}, -p_{ij})$ . In particular,

$$H_0 = \sum_{i=1}^n H_{i0}, \quad H_1 = \sum_{i=1}^n \sum_{j=1}^{n_i} H_{ij}, \quad H_{ij} = H_{ij}(I_{ij}, q_{ij}, p_{ij})$$

where, from now on, we use the same notation for a function and its lift by misuse of language. We will assume that each coordinate cylinder  $S^1 \times (a_{ij}, b_{ij}) \times \{(0, 0)\} \subset M_{ij}$  consists of *relative equilibrium points*, i.e. the co-vector  $dH_{ij}$  is proportional to  $dI_{ij}$  at any its point (with coefficient depending on the point),  $1 \leq i \leq n$ ,  $0 \leq j \leq n_i$ . Therefore the  $2N$ -dimensional symplectic submanifold

$$\prod_{i=1}^n \prod_{j=0}^{n_i} S^1 \times (a_{ij}, b_{ij}) \times \{(0, 0)\} \subset M_0 \times M_1 \quad (25)$$

is invariant under the flow of the model system (24) and it is fibred by invariant  $N$ -dimensional tori  $\prod_{i=1}^n \prod_{j=0}^{n_i} S^1 \times \{(I_{ij}, 0, 0)\}$  where  $N := \sum_{i=1}^n (1 + n_i)$ . Those

solutions of the model system whose phase orbits are contained in the invariant submanifold (25) will be called the *generating solutions*. Consider one of these  $N$ -dimensional tori,  $\Lambda^\circ$ , and a  $(4N - 2)$ -dimensional “transversal surface”  $\Sigma$  in the  $4N$ -dimensional phase space  $M_0 \times M_1$ , which is transversal to the two-dimensional phase orbits of the generating solutions contained in the  $N$ -torus  $\Lambda^\circ$ .

Let us describe the *unperturbed system*. Suppose that a  $S^1$ -invariant function  $F_{ij} = F_{ij}(I_{i0}, q_{i0}, p_{i0}, \varphi_{ij} - \varphi_{i0}, I_{ij}, q_{ij}, p_{ij})$  is given on each direct product  $M_{i0} \times M_{ij}$ ,  $1 \leq j \leq n_i$ . Put

$$\Phi := \sum_{i=1}^n \sum_{j=1}^{n_i} F_{ij}. \quad (26)$$

As the *unperturbed system*, we will regard the 0-Hamiltonian system

$$(M_0 \times M_1, M_0, p; \omega_0, \omega_1; \omega H_0, H_1 + \omega^2 \Phi)^0 \quad (27)$$

with parameter  $0 < \omega \ll 1$ . Then the  $S^1$ -action is given via the Hamiltonian function  $I_0 = \sum_{i=1}^n I_{i0}$  and the 0-Hamiltonian function  $I_1 = \sum_{i=1}^n \sum_{j=1}^{n_i} I_{ij}$ .

Both systems described above: the model one (24) and the unperturbed one (27), are systems with slow and fast variables, because of a small factor  $\omega$  in their Hamiltonian function  $\omega H_0$ .

**THEOREM 2.1 (THE NUMBER OF RELATIVELY-PERIODIC SOLUTIONS).** *Suppose that all solutions of each Hamiltonian system  $(M_{ij}, \widehat{\omega}_{ij}, H_{ij})$  are periodic with periods  $T_{ij} \circ H_{ij}$ , for some functions  $T_{ij} = T_{ij}(h) \neq 0$ . Moreover the number of the systems is  $N \geq 2$ . Suppose that every function  $H_{ij} = H_{ij}(I_{ij}, q_{ij}, p_{ij})$  satisfies the following conditions at all points  $(I_{ij}, 0, 0) \in (a_{ij}, b_{ij}) \times \{(0, 0)\}$ :*

$$dH_{ij} = \Omega_{ij}(I_{ij})dI_{ij}, \quad \frac{\partial^2 H_{ij}}{\partial I_{ij}^2} \neq 0, \quad \det \frac{\partial^2 H_{ij}}{\partial (q_{ij}, p_{ij})^2} = \Omega_{ij}^2(I_{ij}) \quad (28)$$

where  $\Omega_{ij}(I_{ij}) := 2\pi/T_{ij}(H_{ij}(I_{ij}, 0, 0))$ ,  $1 \leq i \leq n$ ,  $0 \leq j \leq n_i$ .

Then, for any collection of segments  $[a_{ij}^\circ, b_{ij}^\circ] \subset (a_{ij}, b_{ij})$ , there exist real numbers  $\omega_0, C_1, C_2 > 0$  such that the following conditions hold for any  $\omega \in (0, \omega_0]$ . Suppose that, for some numbers  $I_{ij}^\circ \in [a_{ij}^\circ, b_{ij}^\circ]$ , the collection of “angular frequencies”  $\Omega_{ij} = \Omega_{ij}(I_{ij}^\circ)$  satisfies the following two conditions. The first condition is the “relative resonance” condition:

$$\omega_1 \overline{l} \Omega_{ij} = \omega_1 + k_{ij} \frac{2\pi}{T}, \quad 1 \leq i \leq n, \quad 0 \leq j \leq n_i, \quad (29)$$

where integers  $k_{ij} \in \mathbb{Z}$  do not vanish simultaneously,  $\overline{l} := \max\{0, \ell\}$ ,  $\omega_1 := \omega \Omega_{10}$  and  $T > 0$ . The second condition is either the nondegeneracy condition:

$$\alpha := \omega_1 T \notin [-C_1 \omega^2 T, C_1 \omega^2 T] + 2\pi\mathbb{Z}, \quad (30)$$

or the following more delicate condition:

$$\alpha \notin 2\pi\mathbb{Z}, \quad \alpha + \Delta_{ij} \omega^2 T \notin [-C_2 \omega^3 T, C_2 \omega^3 T] + 2\pi\mathbb{Z} \quad (31)$$

for all  $1 \leq i \leq n$ ,  $1 \leq j \leq n_i$ . Here the real number  $\Delta_{ij}$  is expressed via the first and second partial derivatives of the function  $\langle F_{ij}^\circ \rangle = \langle F_{ij}^\circ \rangle(q_{ij}, p_{ij})$  at the point  $(0, 0)$ , for instance

$$\Delta_{ij} := \frac{\Omega_{ij}}{2} \text{Tr} \left( \left( \frac{\partial^2 H_{ij}(I_{ij}^\circ, 0, 0)}{\partial (q_{ij}, p_{ij})^2} \right)^{-1} \frac{\partial^2 \langle F_{ij}^\circ \rangle(0, 0)}{\partial (q_{ij}, p_{ij})^2} \right) \quad \text{if } d\langle F_{ij}^\circ \rangle(0, 0) = 0. \quad (32)$$

Here the function  $\langle F_{ij}^\circ \rangle = \langle F_{ij}^\circ \rangle(q_{ij}, p_{ij})$  is obtained by averaging the function  $F_{ij}^\circ := F_{ij}|_{\{(I_{i0}^\circ, 0, 0)\} \times H_{ij}^{-1}(H_{ij}(I_{ij}^\circ, 0, 0))}$  along  $\frac{2\pi}{\Omega_{ij}}$ -periodic solutions of the system  $(M_{ij}, \tilde{\omega}_{ij}, H_{ij})$ ,  $C_1 > |\Delta_{ij}|$ ,  $1 \leq i \leq n$ ,  $1 \leq j \leq n_i$ . Then there exists  $\varepsilon_0 > 0$  such that, for  $0 < \varepsilon \leq \varepsilon_0$ , any  $S^1$ -symmetric (“perturbed”) Hamiltonian system

$$(M, \omega_0 + \varepsilon \omega_1, \tilde{H}) \cong (M, M_0, p; \omega_0, \omega_1; \tilde{H}, \tilde{H}_1 + \omega^2 \tilde{\Phi})^\varepsilon \quad (33)$$

on  $M = M_0 \times M_1$  has at least  $N - 1$   $(T, \alpha)$ -periodic solutions close to generating solutions with angular frequencies  $\omega \Omega_{i0}, \Omega_{ij}$ , provided that  $\tilde{H} := \omega \tilde{H}_0 + \varepsilon \tilde{H}_1 + \omega^2 \varepsilon \tilde{\Phi}$ , the function  $\tilde{H}_0$  “projects” to the factor  $M_0$ , and  $\|\tilde{H}_0 - H_0\|_{C^2} + \|\tilde{H}_1 - H_1\|_{C^2} + \|\tilde{\Phi} - \Phi\|_{C^2} \leq \varepsilon_0$ . Moreover there exist at least  $2^{N-2}$  such solutions counted with multiplicities. The phase orbits of these solutions are contained in some  $N$ -dimensional torus  $\tilde{\Lambda}$  that is  $O(\omega)$ -close (and even  $O(\omega^2)$ -close in the case  $d\langle F_{ij}^\circ \rangle(0, 0) = 0$ ) to the torus  $\Lambda^\circ := \prod_{i=1}^n \prod_{j=0}^{n_i} S^1 \times \{(I_{ij}^\circ, 0, 0)\}$  with respect to a  $C^1$ -norm. The intersection points of these phase orbits with the transversal surface  $\Sigma$  (i.e. with the transversal section to the two-dimensional phase orbits of generating solutions on  $\Lambda^\circ$ ) coincide with critical points of the function  $\Psi|_{\tilde{\Lambda} \cap \Sigma}$  where  $\Psi$  is the generating function of the perturbed succession map  $g_{\tilde{H}, \tilde{H}_1 + \omega^2 \tilde{\Phi}}^T g_{\tilde{I}, I_1}^{-\alpha} : M \rightarrow M$ ,  $\tilde{I} := I_0 + \varepsilon I_1$  (see definition 2.1 below).

Similarly to remark 1.2 (A) one can show that, for any collection  $(\mathbf{a}^\circ, \mathbf{b}^\circ)$  of real numbers  $a_{ij}^\circ, b_{ij}^\circ$  under consideration and for small enough  $0 < \omega \ll 1$ , the period  $T$  can take an arbitrary value of the form  $T \geq 2\pi a_0(\mathbf{a}^\circ, \mathbf{b}^\circ)/\omega$ , hence the quantity  $\omega^2 T$  can be arbitrarily small. Thus any of the nondegeneracy conditions (30) and (31) can always be fulfilled.

**THEOREM 2.2 (SYMMETRIC RELATIVELY-PERIODIC SOLUTIONS).** *Suppose that, under the hypothesis of theorem 2.1, each of three systems: the model system (24), the unperturbed system (27) and the perturbed system (33) is reversible. Then the perturbed system (33) admits exactly  $2^{N-2}$  symmetric  $(T, \alpha)$ -periodic solutions that are  $O(\omega)$ -close (and even  $O(\omega^2)$ -close in the case  $d\langle F_{ij}^\circ \rangle(0, 0) = 0$ ) to the generating symmetric solutions with the angular frequencies under consideration. Each of these  $2^{N-2}$  solutions smoothly depends on the pair of parameters  $(T, \alpha)$ . Moreover the function  $\Psi|_{\tilde{\Lambda}}$  is an even function in the collection  $\varphi$  of angular frequencies  $\varphi_{ij}|_{\tilde{\Lambda}}$ , and the phase orbits of the symmetric  $(T, \alpha)$ -periodic solutions pass through the points  $\varphi$  of the torus  $\tilde{\Lambda}$  with the property  $\varphi = -\varphi$ .*

**THEOREM 2.3 (STABLE PERIODIC SOLUTION).** *Under the hypothesis of theorem 2.1, suppose that all the numbers  $\frac{\partial^2 H_{ij}}{\partial I_{ij}^2}(I_{ij}^\circ, 0, 0)$  have the same sign, e.g. negative (respectively positive), and at least one of the following two conditions holds. The*

first condition consists of the strong nondegeneracy condition:

$$\alpha \notin [-C_1\omega^2T, C_1\omega^2T] + \pi\mathbb{Z}$$

and of the following condition of having the same sign: all the signs

$$\eta_{ij} := \operatorname{sgn} \left( \Omega_{ij} \operatorname{Tr} \frac{\partial^2 H_{ij}(I_{ij}^\circ, 0, 0)}{\partial (q_{ij}, p_{ij})^2} \right), \quad 1 \leq i \leq n, \quad 0 \leq j \leq n_i, \quad (34)$$

are equal. The second condition is the following more delicate one: for any set of real numbers  $\alpha_{ij} \in \mathbb{R}$ ,  $1 \leq i \leq n$ ,  $0 \leq j \leq n_i$ , such that

$$\alpha_{i0} = \eta_{i0}\alpha, \quad |\alpha_{ij} - \eta_{ij}(\alpha + \Delta_{ij}\omega^2T)| \leq C_2\omega^3T, \quad 1 \leq i \leq n, \quad 1 \leq j \leq n_i,$$

the sum of any two, possibly coinciding, numbers of the set does not belong to the set  $2\pi\mathbb{Z}$ . Then the  $((T, \alpha)$ -periodic due to theorem 2.1) phase orbit of the perturbed system (33) passing through any nondegenerate critical point of local minimum (respectively maximum) of the function  $\Psi|_{\tilde{\Lambda} \cap \Sigma}$  is orbitally structurally stable in linear approximation (see definition 3.1).

In all the next statements of the present section, the nondegeneracy condition (30) is not assumed to be fulfilled.

**THEOREM 2.4 (NECESSITY OF THE NONDEGENERACY CONDITION).** *Suppose that, under the hypothesis of theorem 2.1, the number  $\omega \in (0, \omega_0]$  and the collection of angular frequencies  $\Omega_{ij} = \Omega_{ij}(I_{ij}^\circ, 0, 0)$  satisfy all the conditions apart from the nondegeneracy condition. Suppose that  $\alpha \in 2\pi\mathbb{Z}$  and that a smooth function  $R_0$  on  $M$  “projects” to the factor  $M_0$ . Let us fix the two-dimensional torus  $\gamma \subset \Lambda^\circ$  corresponding to a  $T$ -periodic solution of the model system (24). Suppose that some (and, hence, any) point of  $\gamma$  is not a critical point of the function  $\langle R_0^\circ \rangle$  that is obtained by averaging the function  $R_0^\circ := R_0|_{\cap_i H_{i0}^{-1}(H_{i0}(I_{i0}^\circ, 0, 0))}$  along  $T$ -periodic solutions of the model system. Then, for any real number  $\tilde{D} > 0$ , there exist a real number  $\mu_0 > 0$  and a neighbourhood  $U_0$  of the projection  $p(\gamma)$  of the two-dimensional torus  $\gamma$  to  $M_0$  such that, for any  $\varepsilon > 0$ ,  $\omega\varepsilon/\mu_0 \leq \mu \leq \mu_0$  and  $|\tilde{T} - T| + |\tilde{\alpha}| \leq D\mu$ , the following holds. The neighbourhood  $U := U_0 \times M_1$  of  $\gamma$  does not contain any  $(\tilde{T}, \tilde{\alpha})$ -periodic solution of the perturbed system (33), provided that  $\tilde{H}_0 = H_0 + \mu\tilde{R}_0$ , the function  $\tilde{R}_0$  “projects” to the factor  $M_0$ , the function  $\tilde{H}_1$  “projects” to the factor  $M_1$ ,  $\|\tilde{R}_0 - R_0\|_{C^2} \leq \mu_0$ , and  $\|\tilde{\Phi}\|_{C^2} \leq D$ .*

**2.1. Scheme of the proof of theorems 2.1–2.4.** Let us describe two main stages of the proof of theorems 2.1–2.4.

*Stage one* is based on generalizing the averaging method (which was initially [12, 13, 7] introduced for Hamiltonian systems) to the case of 0-Hamiltonian systems with slow and fast variables. (A similar generalization see in [14].) It studies the unperturbed system (27) taking into account that it is close to a “super-integrable” model system (24). At first, one describes  $(T, \alpha)$ -periodic solutions of the unperturbed system (27) that are close to the generating solutions of the model system (24). At second, one studies the linearization of the “succession map” for such solutions (see theorem 2.5 and corollary 2.1).



As is well known, any symplectic linear operator decomposes into the direct product of “indecomposable” symplectic operators. Each of the latter operators is given by a “standard” symplectic matrix called a *Jordan-Kronecker block* in some canonical (i.e. symplectic) basis.

**THEOREM 2.5 (JORDAN-KRONECKER BLOCKS OF A LINEARIZATION OF THE UNPERTURBED SUCCESSION MAP).** *Suppose that, under the hypothesis of theorem 2.1, the number  $\omega \in (0, \omega_0]$  and the set of angular frequencies  $\Omega_{ij} = \Omega_{ij}(I_{ij}^\circ, 0, 0)$  satisfy all the conditions apart from the nondegeneracy conditions. Then there exists a  $N$ -dimensional torus  $\Lambda$  that is  $O(\omega)$ -close (and even  $O(\omega^2)$ -close in the case of  $d\langle F_{ij}^\circ \rangle(0, 0) = 0$ ) with respect to a  $C^1$ -norm to the torus  $\Lambda^\circ$  and is formed by the phase orbits of  $(T, \alpha)$ -periodic solutions of the unperturbed system (27). Moreover, for any point  $x \in \Lambda$ , there exist canonical frames  $e_{ij1}, e_{ij2}, e_{ij3}, e_{ij4}$  in the tangent spaces  $T_{x_{ij}}M_{ij}$  (where  $x_{ij} := \text{pr}_{ij}(x)$ ,  $\text{pr}_{ij} : M \rightarrow M_{ij}$  is the projection) such that the linear part  $d(g_{\omega H_0, H_1 + \omega^2 \Phi}^{-\alpha} g_{I_0, I_1}^{-\alpha})(x)$  of the unperturbed “succession map” at the point  $x$  with respect to this frame is given by a blockwise lower-triangular matrix with the diagonal blocks*

$$\begin{pmatrix} 1 & \omega^{1-j} T \frac{\partial^2 H_{ij}}{\partial I_{ij}^2}(I_{ij}^\circ, 0, 0) & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos \alpha_{ij} & \sin \alpha_{ij} \\ 0 & 0 & -\sin \alpha_{ij} & \cos \alpha_{ij} \end{pmatrix}, \quad \begin{matrix} 1 \leq i \leq n, \\ 0 \leq j \leq n_i. \end{matrix}$$

Here  $\alpha_{ij}$  are some real numbers such that  $\alpha_{i0} = \eta_{i0}\alpha$ ,  $|\alpha_{ij} - \eta_{ij}(\alpha + \Delta_{ij}\omega^2 T)| \leq C_2\omega^3 T$  for  $1 \leq j \leq n_i$ , while the numbers  $\Delta_{ij}$  and the signs  $\eta_{ij} \in \{1, -1\}$  are the same as in (32) and (34). Furthermore all non-diagonal blocks vanish, apart from those blocks whose row and line correspond to the factors  $M_{i0}$  and  $M_{ij}$  (respectively) in the direct product  $M = \prod_{i=1}^n \prod_{j=0}^{n_i} M_{ij}$ . The vectors  $e_{ijk}$  are bounded ( $k = 1, 2, 3, 4$ ), and the relations  $e_{ij1} = \partial/\partial\varphi_{ij}$ ,  $e_{ij2} = \partial/\partial I_{ij}$ ,  $e_{ij3} \in \mathbb{R}_{>0} \partial/\partial q_{ij}$  are either exact for  $j = 0$  or hold up to  $O(\omega)$  for  $1 \leq j \leq n_i$ .

In particular,  $\alpha_{ij} \not\equiv 0 \pmod{2\pi}$  for any  $i, j$  if the nondegeneracy condition holds;  $\alpha_{ij} \not\equiv 0 \pmod{\pi}$  if the strong nondegeneracy condition holds.

Let us explain why the matrix in theorem 2.5 is a blockwise lower-triangular matrix. Unlike to the model system (24), which is a direct product, the unperturbed system (27) is only a semi-direct product. This is why, in theorem 2.5, the linear part of the unperturbed succession map  $g_{\omega H_0, H_1 + \omega^2 \Phi}^{-\alpha} g_{I_0, I_1}^{-\alpha}$  is given by a blockwise lower-triangular matrix, but not by a blockwise diagonal matrix. Notice that the diagonal blocks in theorem 2.5 have the form  $\exp(\omega^{1-j} T V_{ij})$  where

$$V_{ij} = \begin{pmatrix} 0 & \frac{\partial^2 H_{ij}}{\partial I_{ij}^2}(I_{ij}^\circ, 0, 0) & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \widehat{\Omega}_{ij} \\ 0 & 0 & -\widehat{\Omega}_{ij} & 0 \end{pmatrix}, \quad \begin{matrix} 1 \leq i \leq n, \\ 0 \leq j \leq n_i, \end{matrix} \quad (35)$$

$\widehat{\Omega}_{i0} = \eta_{i0}\Omega_{i0}$ ,  $|\widehat{\Omega}_{ij} - \eta_{ij}(\Omega_{ij} + \Delta_{ij}\omega^2)| \leq C_2\omega^3$  and  $\alpha_{ij} = \widehat{\Omega}_{ij}T \pmod{2\pi}$  for  $1 \leq j \leq n_i$ .

From theorem 2.5, one can easily derive the following its refinement.

COROLLARY 2.1. *Suppose that, under the hypothesis of theorem 2.5, the model and the unperturbed systems (24) and (27) are reversible. Then a  $((T, \alpha)$ -periodic by theorem 2.5) solution  $\gamma(t)$  of the unperturbed system with  $\gamma(0) \in \Lambda$  is symmetric if its initial point  $\gamma(0) =: \{\varphi_{ij}, I_{ij}, q_{ij}, p_{ij}\}_{i,j}$  satisfies the conditions  $\varphi_{ij} = -\varphi_{ij} \pmod{2\pi}$ . Hence the points  $x := \gamma(0)$  and  $x' := g_{I_0, I_1}^{-\alpha/2}(\gamma(T/2))$  are fixed under the involution  $J$ . Moreover, there exists a unique tangent frame  $e(x) = \{e_{ij1}, e_{ij2}, e_{ij3}, e_{ij4}\}_{i,j}$  at the point  $x$  that satisfies the conditions of theorem 2.5 and consists of eigenvectors of the involution  $dJ(x)$  with eigenvalues  $\{-1, 1, 1, -1\}_{i,j}$ . In the frames  $e(x)$  and  $e(x')$ , the operator  $A' = A'(x) = d(g_{\omega_{H_0, H_1} + \omega^2 \Phi}^{-\alpha/2})_{I_0, I_1}(x) : T_x M \rightarrow T_{x'} M$  is given by a blockwise lower-triangular matrix that is analogous to the matrix in theorem 2.5 and has the diagonal blocks  $\exp((\omega^{1-j} T/2) V_{ij})$ , see (35).*

Stage two of the proof of theorems 2.1–2.4 is based on generalizing the “method of generating function” (which was initially [4, 13, 15, 7, 16] introduced for Hamiltonian systems) to the case of an  $\varepsilon$ -Hamiltonian (“perturbed”) system that is  $C^2$ -close to a 0-Hamiltonian (“unperturbed”) system. It studies  $T$ -periodic trajectories of the perturbed system in a neighbourhood of a “nondegenerate” compact submanifold  $\Lambda$  formed by the phase trajectories of  $T$ -periodic solutions of the unperturbed system. At first, one proves that the intersections points of  $T$ -periodic trajectories of the perturbed system with the “transversal surface”  $\Sigma$  (see the formulation of theorem 2.1) coincide with critical points of the function  $\Psi|_{\Sigma \cap \tilde{\Lambda}}$ . At second, one studies the linearization of the “succession map” for these solutions. Here  $\Psi$  is the generating function of the perturbed succession map (see definition 2.1), and  $\tilde{\Lambda}$  is a submanifold that is  $C^1$ -close to the submanifold  $\Lambda$ . One proves the fact from above similarly to the case when both systems (the unperturbed and the perturbed ones) are Hamiltonian [13, 7]. In more detail, one constructs the submanifold  $\tilde{\Lambda}$  in the same way (since the construction does not use that the systems are Hamiltonian). Further, one proves that the submanifold  $\Lambda$  is nondegenerate and that critical points of the function  $\Psi|_{\Sigma \cap \tilde{\Lambda}}$  coincide with critical points of the perturbed “succession map” via the results of the first stage.

DEFINITION 2.1 (GENERATING FUNCTION). Let  $\varepsilon > 0$  and  $A : M \rightarrow M$  be a symplectic self-map of a symplectic manifold  $(M, \omega) = (M_0 \times M_1, \omega_0 + \varepsilon \omega_1)$ . Denote  $v_{ij1} := \varphi_{ij}$ ,  $v_{ij2} := q_{ij}$  (“coordinates”),  $u_{ij1} := I_{ij}$ ,  $u_{ij2} := p_{ij}$  (“impulses”). Denote by  $\alpha$  the differential 1-form of the type of  $(\mathbf{u}' - \mathbf{u})d\mathbf{v} + (\mathbf{v} - \mathbf{v}')d\mathbf{u}'$  on  $M$ . More precisely, we define  $\alpha$  by the formula

$$\alpha(x) := \sum_{i=1}^n \sum_{k=1}^2 \left( (u'_{i0k} - u_{i0k}) dv_{i0k} + (v_{i0k} - v'_{i0k}) du'_{i0k} + \right. \\ \left. + \varepsilon \sum_{j=1}^{n_i} ((u'_{ijk} - u_{ijk}) dv_{ijk} + (v_{ijk} - v'_{ijk}) du'_{ijk}) \right)$$

where  $A : x = (\mathbf{v}, \mathbf{u}) \mapsto A(x) =: (\mathbf{v}', \mathbf{u}')$ . In other words,

$$\alpha(x)\xi := \omega(A(x) - x, dP(x)\xi), \quad \xi \in T_x M, \quad (36)$$

where  $P : x = (\mathbf{v}, \mathbf{u}) \mapsto P(x) := (\mathbf{v}, \mathbf{u}')$ , and  $\omega(\xi, \eta)$  denotes the value of the symplectic structure  $\omega = \sum_{i=1}^n \sum_{k=1}^2 (du_{i0k} \wedge dv_{i0k} + \varepsilon \sum_{j=1}^{n_i} du_{ijk} \wedge dv_{ijk})$  on the

pair of vectors  $\xi, \eta \in T_x M$ . A function  $\Psi = \Psi(x)$  will be called the *generating function* of the map  $A$  if it satisfies the condition

$$d\Psi(x) = \alpha(x), \quad x \in M. \quad (37)$$

Let us show that such a function  $\Psi$  exists, i.e. the form  $\alpha$  is exact. One easily shows that the integral of the form  $\alpha$  along any closed curve equals the integral of the symplectic structure  $\omega$  along some two-dimensional torus. The latter integral vanishes, since the symplectic structure under consideration is exact (being the standard symplectic structure on  $M = T^*Q$ ). Thus, the integral of the form  $\alpha$  along any closed curve vanishes. This proves that the function  $\Psi$  is well-defined up to an additive constant.

### § 3. Normalizing the $N + 1$ body problem of the type of planetary system with satellites

Consider the problem about the motion of a system of  $N + 1$  particles attracting by Newton's law on a Euclidean plane  $E^2$ ,  $N \geq 2$ . The particles attract with the Newtonian potential  $U$  from (2). The equations of the motion have the form (1).

The configuration manifold of the problem under consideration consists of all sets of radius vectors  $\mathbf{r}_i \in E^2$ ,  $0 \leq i \leq N$ , such that  $\mathbf{r}_i \neq \mathbf{r}_j$ ,  $0 \leq i < j \leq N$ . In particular, the planar  $N + 1$  body problem has  $2N + 2$  degrees of freedom.

The system of equations of the motion in the cotangent bundle of the configuration manifold is a Hamiltonian system with the Hamiltonian function  $H$  being the total energy of the system:

$$H = G + U \quad \text{where} \quad G = \sum_{i=0}^N \frac{\mathbf{p}_i^2}{2\mu_i}, \quad U = - \sum_{0 \leq i < j \leq N} \frac{g\mu_i\mu_j}{r_{ij}}, \quad (38)$$

and the symplectic structure

$$\omega = d\mathbf{p} \wedge d\mathbf{q} = \sum_{i=0}^N (dp_i^1 \wedge dq_i^1 + dp_i^2 \wedge dq_i^2).$$

Here  $\mathbf{q} = (\mathbf{r}_0, \dots, \mathbf{r}_N) = (r_0^1, r_0^2, \dots, r_N^1, r_N^2)$  are coordinates and  $\mathbf{p} = (\mathbf{p}_0, \dots, \mathbf{p}_N) = (p_0^1, p_0^2, \dots, p_N^1, p_N^2)$  are impulses (i.e. the conjugate variables to the variables  $\mathbf{q}$ ), while  $G$  and  $U$  are the kinetic energy and the potential energy (2) of the system.

**3.1. Reducing to a system with  $2N$  degrees of freedom.** Consider the “reduced” problem, which is obtained by transferring to the coordinate system centred at the centre of masses

$$\mathbf{c} = \mathbf{c}(\mathbf{r}_0, \dots, \mathbf{r}_N) = \frac{\sum_{i=0}^N \mu_i \mathbf{r}_i}{\sum_{i=0}^N \mu_i}$$

of all particles. The configuration manifold  $Q$  of the obtained problem is an open subset of the  $2N$ -dimensional vector space  $\{(\mathbf{r}_0, \dots, \mathbf{r}_N) \in (E^2)^{N+1} \mid \mathbf{c}(\mathbf{r}_0, \dots, \mathbf{r}_N) = 0\}$ :

$$Q = \{(\mathbf{r}_0, \dots, \mathbf{r}_N) \in (E^2)^{N+1} \mid \mathbf{c}(\mathbf{r}_0, \dots, \mathbf{r}_N) = 0, \mathbf{r}_i \neq \mathbf{r}_j, i < j\}. \quad (39)$$

The variables  $\mathbf{q}' = (\mathbf{r}_1, \dots, \mathbf{r}_N)$  are taken as coordinates on  $Q$ .

As a phase space of the problem, we take the cotangent bundle  $X := T^*Q$  to the configuration manifold. The phase space is  $4N$ -dimensional. Its natural coordinates are the variables  $\mathbf{q}'$  (coordinates) and their conjugates  $\mathbf{p}'$  (impulses). From now on, the accents will be omitted.

**3.2. Stating the problem.** Our general problem is to find  $(T, \alpha)$ -periodic solutions of the system described above, see definition 1.1.

We observe that the motions with respect to the rotating coordinate system with angular velocity  $\omega_1$  are described by the Hamiltonian system with the Hamiltonian function  $H - \omega_1 I$  where

$$I := \sum_{i=0}^N [\mathbf{q}_i, \mathbf{p}_i] \quad (40)$$

is the “integral of areas”, also called the *angular momentum* [17]. Here  $[\mathbf{q}, \mathbf{p}]$  is the oriented area of the parallelogram spanned by the vectors  $\mathbf{q}$  and  $\mathbf{p}$ . Thus the problem is equivalent to finding  $T$ -periodic solutions of the Hamiltonian system with the Hamiltonian function  $H - \omega_1 I$  where  $\omega_1 = \frac{\alpha - 2\pi k}{T}$ ,  $k$  being any integer. Due to (9) and (10), we can define  $\omega_1$  to be the angular frequency of any of the planets or satellites, for example the angular frequency of the first planet.

**3.2.1. The parameters of the problem.** Let us enumerate the particles of the system in a more convenient way (see (3) and (4)). Namely, we assume that one of the particles  $\mathbf{r}_0$  (the Sun) has mass  $\mu_0 = 1$ , and the masses of the other particles ( $n$  planets  $\mathbf{r}_i$  and  $N - n$  satellites  $\mathbf{r}_{ij}$ ) have the form (3) where  $0 < \mu, \nu \ll 1$  are small parameters and  $m_i, m_{ij} > 0$ ,  $1 \leq i \leq n$ ,  $1 \leq j \leq n_i$ , are the parameters of the form (4). We will say that the  $i$ th planet and all its  $n_i$  satellites form the  $i$ th *satellite system*.

As above (see (8)), let us introduce “relative coordinates” in the configuration space  $Q$  (see (39)). Namely, we draw radius vectors

$$\mathbf{y}_{ij} := \mathbf{r}_{ij} - \mathbf{r}_i, \quad 1 \leq j \leq n_i, \quad (41)$$

from each planet to its satellites (if any). We draw also the “scaled” radius vector

$$\mathbf{x}_i := (\mathbf{c}_i - \mathbf{r}_0)/R, \quad 1 \leq i \leq n, \quad (42)$$

from the sun  $\mathbf{r}_0$  to the centre of masses  $\mathbf{c}_i$  of the  $i$ th satellite system where

$$\mathbf{c}_i := \left( m_i \mathbf{r}_i + \nu \sum_{j=1}^{n_i} m_{ij} \mathbf{r}_{ij} \right) / \left( m_i + \nu \sum_{j=1}^{n_i} m_{ij} \right).$$

**3.2.2. Symmetric solutions.** Let us show that conditions 1 and 2 in definition 1.2 of a symmetric solution are equivalent. For this we will use the invariance of the total energy  $H$  of the system under the following two involutions  $S_l$  and  $S$  in the phase space  $T^*Q$ .

Let  $l$  be any line in the plane of the motion going through the centre of masses of the system of particles. Define the following three transformations in the phase space  $T^*Q$  preserving the total energy  $H$  of the system:

1) the canonical involution  $S_l : T^*Q \rightarrow T^*Q$  corresponding to the axial symmetry, i.e. to the self-map of the configuration manifold  $Q$  sending all particles of the system to their images under the symmetry with respect to the line  $l$ ;

2) the anti-canonical involution  $S$  (“reversion of time”) sending each pair  $(\mathbf{q}, \mathbf{p}) \in T^*Q$  to the pair  $(\mathbf{q}, -\mathbf{p})$  where  $\mathbf{q}$  and  $\mathbf{p}$  are the sets of “coordinates” and “impulses” of all particles of the system;

3) the anti-canonical involution  $J_l := SS_l = S_lS$ .

Each of these transformations is an involution, i.e. coincides with its inverse. The first involution is canonical, i.e. preserves the canonical symplectic structure  $d\mathbf{p} \wedge d\mathbf{q}$  on  $T^*Q$ . The second and the third involutions are anti-canonical, i.e. they affect the symplectic structure by changing its sign to the opposite. Thus all three involutions move trajectories of the system to trajectories, moreover the first involution preserves the time on trajectories, while the second and the third involutions “reverse the time”.

A solution satisfies the first (respectively the second) condition of symmetry if and only if the point of the phase space corresponding to the time  $t_0$  (respectively any time  $t \in \mathbb{R}$ ) of this solution is fixed (respectively is mapped to the point corresponding to the time  $2t_0 - t$ ) under the involution  $J_l = SS_l = S_lS$ . This shows the equivalence of the conditions 1 and 2 in definition of a symmetric solution.

A solution  $\gamma(t)$  is symmetric and  $(T, \alpha)$ -periodic if and only if its points  $\gamma(t_0)$  and  $\gamma(t_0 + T/2)$  are fixed under the involutions  $J_l$  and  $J_{R_{\alpha/2}(l)}$  respectively where  $R_\varphi : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  denotes the rotation by the angle  $\varphi$ .

**3.2.3. A stable relatively-periodic solution.** Suppose that a Hamiltonian system  $(X, \omega, H)$  is  $S^1$ -symmetric with respect to the Hamiltonian action of a circle  $S^1$  on  $X$  via the Hamiltonian function  $I$ . Then the function  $I$  is a first integral of the system. Consider the flow  $g_{H-\omega_1 I}^t : X \rightarrow X$ ,  $t \in \mathbb{R}$ , of the system with the Hamiltonian function  $H - \omega_1 I$ . The map  $A := g_{H-\omega_1 I}^T$  will be called the *succession map*, and its linear part  $dA(x)$  at a fixed point  $x$  will be called the *monodromy operator* at this point.

Let us define a “reduced” succession map for the two-dimensional torus  $\gamma$  corresponding to a  $(T, \omega_1 T)$ -periodic solution. Let  $\Sigma \subset X$  be a small surface of codimension 2 that transversally intersects the torus  $\gamma$  at some point  $x_0 = \gamma \cap \Sigma$ . Consider the restriction of the system to a regular common level set

$$X_{H,I} := \{H = \text{const}, I = \text{const}\} \supset \gamma$$

of the first integrals  $H$  and  $I$ . Consider the small surface  $\sigma := \Sigma \cap X_{H,I}$  of codimension 2 in  $X_{H,I}$ , which transversally intersects the torus  $\gamma$  at the point  $x_0 = \gamma \cap \sigma$ . Consider the two-dimensional foliation on the manifold  $X_{H,I}$  whose fibres are invariant under the (commuting) flows of the systems with Hamiltonian functions  $H$  and  $I$ ; this condition uniquely determines fibres. Take the self-map  $\bar{A}$  of the surface  $\sigma$  sending any point  $x \in \sigma$  to the “next intersection point” of the fibre containing the point  $x$  with the surface  $\sigma$ . In more detail, the map  $\bar{A} : \sigma' \rightarrow \sigma$  is defined in a sufficiently small neighbourhood  $\sigma' \subset \sigma$  of the point  $x_0$  in  $\sigma$ , it is “close” to the map  $A|_{\sigma'} = g_{H-\omega_1 I}^T|_{\sigma'}$  and has the form  $\bar{A}(x) = g_{H-f_1(x)I}^{f_2(x)}(x)$ . Here  $f_1$  and  $f_2$  are some smooth functions on  $\sigma'$  defined by the conditions  $\bar{A}(x) \in \sigma$ ,  $x \in \sigma'$ ,  $f_1(x_0) = \omega_1$ ,  $f_2(x_0) = T$ . The map  $\bar{A}$  will be called the *reduced succession map* (or

the *Poincaré map*), and its linear part  $\mathbf{A} = d\bar{A}(x_0)$  at the point  $x_0$  will be called the *reduced monodromy operator* corresponding to the torus  $\gamma$ .

Recall that a linear operator  $\mathbf{A}$  is called (*Liapunov*) *stable* if the norm of the operator  $\mathbf{A}^n$  is bounded for  $n \rightarrow \infty$ . As is well-known [17], the transversal surface  $\sigma$  is a symplectic submanifold and the self-map  $\bar{A}$  of this surface is also symplectic. In particular, the reduced monodromy operator  $\mathbf{A}$  is symplectic too. A symplectic operator  $\mathbf{A}$  is called *structurally stable* if it is stable and any symplectic operator that is close enough to  $\mathbf{A}$  is stable too.

**DEFINITION 3.1.** The two-dimensional torus  $\gamma$  and the corresponding relatively-periodic solution will be called *orbitally structurally stable in linear approximation (OSSL)* (respectively *orbitally stable in linear approximation on a common level surface of the first integrals of energy and angular momentum (OSLI)*) if the reduced monodromy operator  $\mathbf{A} = d\bar{A}(x_0)$  corresponding to the torus  $\gamma$  is structurally stable (respectively stable). The torus  $\gamma$  will be called *isoenergetically nondegenerate (IN)* if 1 does not belong to the spectrum of the reduced monodromy operator  $\mathbf{A}$ , i.e.  $1 \notin \text{spec } \mathbf{A}$ . The torus  $\gamma$  is called *orbitally stable in linear approximation (OSL)* if the linear operator  $\mathbf{B} = d\bar{B}(x_0)$  is stable, where  $\bar{B} : \Sigma' \rightarrow \Sigma$  is the map defined similarly to the Poincaré map  $\bar{A} : \sigma' \rightarrow \sigma$ .

**DEFINITION 3.2.** An eigenvalue  $\lambda \in \mathbb{C}$  of a symplectic operator  $\mathbf{A}$  is called *elliptic* [17] if it satisfies one of the following equivalent conditions:

1) the quadratic form  $Q(\xi) = \omega(\mathbf{A}\xi, \xi)$  is (positive or negative) definite on the maximal invariant subspace where the operator  $\mathbf{A}$  has no eigenvalues apart from  $\lambda$  and  $\bar{\lambda}$ ;

2) the Hermitian quadratic form  $\frac{1}{2i}\omega(\xi, \bar{\xi})$  is (positive or negative) definite on the complex eigensubspace with eigenvalue  $\lambda$  of the complexified space.

The quadratic form  $Q$  is called *the generating function* of the symplectic operator  $\mathbf{A}$  (see also definition 2.1).

**PROPOSITION 3.1.** (A) *A symplectic operator is stable if and only if it is diagonalizable over  $\mathbb{C}$  and all its eigenvalues belong to the unit circle in  $\mathbb{C}$ .*

(B) *A symplectic operator is structurally stable if and only if all its complex eigenvalues are elliptic.*  $\square$

Let us mention some relations between the stability properties introduced above of an invariant two-dimensional torus  $\gamma$ :

1) the following implications hold:  $\text{IN} \Leftarrow \text{OSSL} \Rightarrow \text{OSL} \Rightarrow \text{OSLI}$ . (The second implication is an important property of tori having the OSSL property; it follows from property 3 below. The first implication follows from proposition 3.1(B). The third implication is obvious. The inverse implications are in general false);

2) if all eigenvalues of a symplectic operator  $\mathbf{A}$  are pairwise different and lie on the unit circle in  $\mathbb{C}$  then it is structurally stable, thus the torus  $\gamma$  is OSSL;

3) if the torus  $\gamma$  is isoenergetically nondegenerate (IN) then it is included into a smooth two-parameter family of isoenergetically nondegenerate two-dimensional tori  $\gamma_{H,I}$  where parameters of the family are values of the first integrals  $H$  and  $I$ . If the torus  $\gamma$  is OSSL (and, hence, IN) then all the invariant tori of this family are also OSSL. Hence  $\text{OSSL} \Rightarrow \text{OSL}$  (and not only OS LI).

We stress that, if the torus  $\gamma$  is OSLI and IN, then the other tori of the family do not need to be OSLI, thus the torus  $\gamma$  does not need to be OSL.

Thus, among the stability properties introduced in definition 3.1 for a periodic solution, the strongest one is the OSSL property, which is studied in the present paper (see theorems 1.2(B) and 2.3).

**3.3. Describing the problem about planetary system with satellites via an  $\varepsilon$ -Hamiltonian system.** We define a smooth function  $F = F(\mathbf{x}, \mathbf{y})$  on  $(\mathbb{R}^2 \setminus \{0\}) \times \mathbb{R}^2$  (or on  $(\mathbb{R}^3 \setminus \{0\}) \times \mathbb{R}^3$ ) by the formula

$$F(\mathbf{x}, \mathbf{y}) = F_{00}(\mathbf{x}, \mathbf{y}) := \frac{\mathbf{x}^2 \mathbf{y}^2 - 3\langle \mathbf{x}, \mathbf{y} \rangle^2}{2|\mathbf{x}|^5}. \quad (43)$$

The function  $F$  will be called *the Hill potential* (or the “limit potential of the action of the Sun to a satellite”). As lemma 3.1 below shows, the unperturbed field of accelerations on the configuration manifold  $Q$  (which can be obtained in the limit when  $\mu, \nu \rightarrow 0$  and  $\omega > 0$  is fixed) is described by the following system of equations in the coordinates (41), (42):

$$\begin{cases} \frac{d^2 \mathbf{x}_i}{dt^2} = -\omega^2 \frac{\mathbf{x}_i}{|\mathbf{x}_i|^3}, & 1 \leq i \leq n, \\ \frac{d^2 \mathbf{y}_{ij}}{dt^2} = -m_i \frac{\mathbf{y}_{ij}}{|\mathbf{y}_{ij}|^3} - \omega^2 \frac{\partial F}{\partial \mathbf{y}}(\mathbf{x}_i, \mathbf{y}_{ij}), & 1 \leq j \leq n_i. \end{cases} \quad (44)$$

For given  $i, j$ , the first equation of the system (44) describes the Kepler problem for the  $i$ th planet, while the second equation describes the so-called *Hill problem* for its  $j$ th satellite when the planet moves in a given way by virtue of the first equation.

REMARK 3.1. (A) The Hill potential  $F(\mathbf{x}, \mathbf{y})$  is in fact the third coefficient of the power series of the function  $-\frac{1}{|\mathbf{x} + \rho \mathbf{y}|}$  in the variable  $\rho$  at zero:

$$\frac{1}{|\mathbf{x} + \rho \mathbf{y}|} = \frac{1}{\sqrt{\mathbf{x}^2 + 2\rho \langle \mathbf{x}, \mathbf{y} \rangle + \rho^2 \mathbf{y}^2}} =: \frac{1}{|\mathbf{x}|} - \rho \frac{\langle \mathbf{x}, \mathbf{y} \rangle}{|\mathbf{x}|^3} - \rho^2 F_{0,\rho}(\mathbf{x}, \mathbf{y}), \quad (45)$$

$$\begin{aligned} F_{0,\rho}(\mathbf{x}, \mathbf{y}) &= F(\mathbf{x}, \mathbf{y}) - \rho \langle \mathbf{x}, \mathbf{y} \rangle \frac{3\mathbf{x}^2 \mathbf{y}^2 - 5\langle \mathbf{x}, \mathbf{y} \rangle^2}{2|\mathbf{x}|^7} - \\ &- \rho^2 \frac{3\mathbf{x}^4 \mathbf{y}^4 - 30\mathbf{x}^2 \langle \mathbf{x}, \mathbf{y} \rangle^2 \mathbf{y}^2 + 35\langle \mathbf{x}, \mathbf{y} \rangle^4}{8|\mathbf{x}|^9} + \dots, \quad \rho \rightarrow 0. \end{aligned} \quad (46)$$

A more general analytic potential

$$F_{\theta,\rho}(\mathbf{x}, \mathbf{y}) = \theta F_{0,-\theta\rho}(\mathbf{x}, \mathbf{y}) + (1 - \theta) F_{0,(1-\theta)\rho}(\mathbf{x}, \mathbf{y}) \quad (47)$$

appears in the three-body problem (with  $0 < \theta < 1$ ,  $\mu, \omega, \rho > 0$  and (8)) and in the *restricted three-body problem* (with  $\theta = 0$ ,  $\mu, \omega, \rho > 0$  and (8)), see §3.3.2 and (70).

We remark that  $\frac{\partial F}{\partial \mathbf{y}}(\mathbf{x}, \mathbf{y}) = \frac{\mathbf{x}^2 \mathbf{y} - 3\langle \mathbf{x}, \mathbf{y} \rangle \mathbf{x}}{|\mathbf{x}|^5}$ .

(B) The unperturbed system (44) shows that the variables  $\mathbf{x}_i$  and  $\mathbf{y}_{ij}$  are automatically slow and fast variables respectively, provided that  $\omega$  is small.

Let  $\mathbf{x}_i, \boldsymbol{\xi}_i, \mathbf{y}_{ij}, \boldsymbol{\eta}_{ij}$  be the natural coordinates in the space  $X = T^*Q$  corresponding to the coordinates  $\mathbf{x}_i, \mathbf{y}_{ij}$  in the configuration space  $Q$ ,  $1 \leq i \leq n$ ,  $1 \leq j \leq n_i$  (see §3.1, 3.2.1).



Denote

$$\bar{m}_i = m_i + \nu \sum_{j=1}^{n_i} m_{ij}, \quad \tilde{m}_i = \frac{\bar{m}_i}{1 + \mu \bar{m}_i}, \quad \bar{m}_{ij} = \frac{m_{ij}}{m_i}, \quad \tilde{m}_{ij} = \frac{m_{ij} m_i}{m_i + \nu m_{ij}}, \quad (48)$$

the total mass of the  $i$ th satellite system, and the “reduced” masses of planets and satellites. Introduce the following functions on  $T^*Q$ : the Hamiltonian functions

$$\tilde{K}_i = \frac{\xi_i^2}{2\tilde{m}_i} - \frac{\bar{m}_i}{|\mathbf{x}_i|}, \quad \tilde{S}_j^{(i)} = \frac{\eta_{ij}^2}{2\tilde{m}_{ij}} - \frac{m_i m_{ij}}{|\mathbf{y}_{ij}|} \quad (49)$$

of the Kepler problems, the angular momenta

$$I_{i0} = [\mathbf{x}_i, \xi_i], \quad I_{ij} = [\mathbf{y}_{ij}, \eta_{ij}], \quad 1 \leq i \leq n, \quad 1 \leq j \leq n_i \quad (50)$$

of planets and satellites, and the “perturbation functions”

$$K_{ii'} = \langle \xi_i, \xi_{i'} \rangle - \frac{\bar{m}_i \bar{m}_{i'}}{|\mathbf{x}_i - \mathbf{x}_{i'}|}, \quad S_{jj'}^{(i)} = \frac{\langle \eta_{ij}, \eta_{ij'} \rangle}{m_i} - \frac{m_{ij} m_{ij'}}{|\mathbf{y}_{ij} - \mathbf{y}_{ij'}|}, \quad (51)$$

$1 \leq i < i' \leq n$ ,  $1 \leq j < j' \leq n_i$ , of planetary system and satellite systems respectively (corresponding to pair-wise interactions of planets, respectively satellites of the same planet).

As a “perturbation potential”, let us consider the function

$$\tilde{\Phi} = \tilde{\Phi}(\mathbf{x}_*, \mathbf{y}_{**}, \bar{m}_*, \bar{m}_{**}, \mu, \nu, \rho) := \sum_{i=1}^n \bar{m}_i \Phi_i + \mu \sum_{1 \leq i < i' \leq n} \bar{m}_i \bar{m}_{i'} \Phi_{ii'} \quad (52)$$

in the configuration variables  $\mathbf{x}_i, \mathbf{y}_{ij} \in \mathbb{R}^2$  and parameters  $\bar{m}_i, \bar{m}_{ij}, \mu, \nu, \rho \in \mathbb{R}$ . Here the functions  $\Phi_i = \Phi_i(\mathbf{x}_i, \mathbf{y}_{i*}, \bar{m}_{i*}, \nu, \rho)$  and  $\Phi_{ii'} = \Phi_{ii'}(\mathbf{x}_i - \mathbf{x}_{i'}, \mathbf{y}_{i*}, \mathbf{y}_{i'*}, \bar{m}_{i*}, \bar{m}_{i'*}, \nu, \rho)$  are defined by the formulae

$$\nu \rho^2 \Phi_i := \frac{1}{|\mathbf{x}_i|} - \frac{m_i / \bar{m}_i}{|\mathbf{x}_i - \nu \rho \delta_i|} - \nu \sum_{j=1}^{n_i} \frac{m_{ij} / \bar{m}_i}{|\mathbf{x}_i + \rho \mathbf{y}_{ij} - \rho \nu \delta_i|}, \quad (53)$$

$$\begin{aligned} \nu \rho^2 \bar{m}_i \bar{m}_{i'} \Phi_{ii'} &:= \frac{\bar{m}_i \bar{m}_{i'}}{|\mathbf{x}_i - \mathbf{x}_{i'}|} - \sum_{j=1}^{n_i} \frac{\nu m_{ij} m_{ij}}{|\mathbf{x}_i - \mathbf{x}_{i'} + \rho \mathbf{y}_{ij} - \rho \nu (\delta_i - \delta_{i'})|} - \\ &- \frac{m_i m_{i'}}{|\mathbf{x}_i - \mathbf{x}_{i'} - \nu \rho (\delta_i - \delta_{i'})|} - \sum_{j'=1}^{n_{i'}} \frac{\nu m_i m_{i'j'}}{|\mathbf{x}_i - \mathbf{x}_{i'} + \rho \mathbf{y}_{ij'} - \rho \nu (\delta_i - \delta_{i'})|} - \\ &- \sum_{j=1}^{n_i} \sum_{j'=1}^{n_{i'}} \frac{\nu^2 m_{ij} m_{i'j'}}{|\mathbf{x}_i - \mathbf{x}_{i'} + \rho (\mathbf{y}_{ij} - \mathbf{y}_{i'j'}) - \rho \nu (\delta_i - \delta_{i'})|}. \end{aligned} \quad (54)$$

Here

$$\delta_i := \sum_{j=1}^{n_i} \frac{m_{ij}}{\bar{m}_i} \mathbf{y}_{ij} = \sum_{j=1}^{n_i} \bar{m}_{ij} \mathbf{y}_{ij} / (1 + \nu \sum_{j=1}^{n_i} \bar{m}_{ij}) \quad (55)$$

is the radius vector drawn from a planet to the centre of masses of the system of its satellites, multiplied by  $(\sum_{j=1}^{n_i} m_{ij})/\bar{m}_i$ . One easily shows (see (45)) that the function  $\Phi_i$  is analytic in all its variables in the region

$$\left\{ 1 + \nu \sum_{j'=1}^{n_i} \bar{m}_{ij'} \neq 0, \quad |\rho| \left( 1 + |\nu| \sum_{j'=1}^{n_i} |\bar{m}_{ij'}| \right) |\mathbf{y}_{ij}| < |\mathbf{x}_i| \right\}_{j=1}^{n_i}, \quad (56)$$

while the function  $\Phi_{ii'}$  is analytic in all its variables in the region

$$\left\{ \begin{array}{l} 1 + \nu \sum_{j=1}^{n_i} \bar{m}_{ij} \neq 0, \quad 1 + \nu \sum_{j'=1}^{n_{i'}} \bar{m}_{i'j'} \neq 0, \\ |\rho|(1 + |\nu| \sum_{j''=1}^{n_i} |\bar{m}_{ij''}|) |\mathbf{y}_{ij}| < \frac{|\mathbf{x}_i|}{2}, \quad 1 \leq j \leq n_i, \\ |\rho|(1 + |\nu| \sum_{j''=1}^{n_{i'}} |\bar{m}_{i'j''}|) |\mathbf{y}_{i'j'}| < \frac{|\mathbf{x}_{i'}|}{2}, \quad 1 \leq j' \leq n_{i'} \end{array} \right\}. \quad (57)$$

The functions  $\Phi_{ii'}$  are expressed in terms of  $\Phi_1, \dots, \Phi_n$  as follows:

$$\begin{aligned} \Phi_{ii'} &= \frac{m_{i'}}{\bar{m}_{i'}} \Phi_i(\mathbf{x}_i - \mathbf{x}_{i'} + \nu \rho \boldsymbol{\delta}_{i'}, \mathbf{y}_{i*}, \bar{m}_{i*}, \nu, \rho) + \Phi_{i'}(\mathbf{x}_{i'} - \mathbf{x}_i, \mathbf{y}_{i'*}, \bar{m}_{i'*}, \nu, \rho) + \\ &+ \nu \sum_{j'=1}^{n_{i'}} \frac{m_{i'j'}}{\bar{m}_{i'}} \Phi_i(\mathbf{x}_i - \mathbf{x}_{i'} - \rho \mathbf{y}_{i'j'} + \nu \rho \boldsymbol{\delta}_{i'}, \mathbf{y}_{i*}, \bar{m}_{i*}, \nu, \rho). \end{aligned} \quad (58)$$

We set  $\Phi_i := 0$  if  $n_i = 0$  (i.e. the  $i$ th planet has no satellites), and we set  $\Phi_{ii'} := 0$  if  $n_i = n_{i'} = 0$ . If  $n_i = 1$  (i.e. the  $i$ th planet is a double planet,  $\theta_i := \nu m_{i1}/(m_i + \nu m_{i1})$ ) then we have  $\Phi_i = \frac{\theta_i(1-\theta_i)}{\nu} F_{\theta_i, \rho}(\mathbf{x}_i, \mathbf{y}_{i1})$  and

$$\begin{aligned} \Phi_{ii'} &= \Phi_{i'}(\mathbf{x}_{i'} - \mathbf{x}_i, \mathbf{y}_{i'*}, \bar{m}_{i'*}, \nu, \rho) + \frac{\theta_i(1-\theta_i)}{\nu \bar{m}_{i'}/m_{i'}} F_{\theta_i, \rho}(\mathbf{x}_i - \mathbf{x}_{i'} + \nu \rho \boldsymbol{\delta}_{i'}, \mathbf{y}_{i1}) + \\ &+ \theta_i(1-\theta_i) \sum_{j'=1}^{n_{i'}} \frac{m_{i'j'}}{\bar{m}_{i'}} F_{\theta_i, \rho}(\mathbf{x}_i - \mathbf{x}_{i'} - \rho \mathbf{y}_{i'j'} + \nu \rho \boldsymbol{\delta}_{i'}, \mathbf{y}_{i1}). \end{aligned}$$

**LEMMA 3.1** (EQUIVALENCE OF THE  $N+1$  BODY PROBLEM TO AN  $\varepsilon$ -HAMILTONIAN SYSTEM). *Let  $Q$  be the  $2N$ -dimensional vector space formed by all configurations of  $N+1$  particles with masses (3) and the centre of masses at the origin in a Euclidean plane. Define linear coordinates on  $Q$  to be the collection of radius vectors  $\mathbf{x}_i, \mathbf{y}_{ij} : Q \rightarrow \mathbb{R}^2$ ,  $1 \leq i \leq n$ ,  $1 \leq j \leq n_i$  (see (41), (42)). There exists a collection of linear functions  $\boldsymbol{\xi}_i, \boldsymbol{\eta}_{ij} : Q^* \rightarrow \mathbb{R}^2$ ,  $1 \leq i \leq n$ ,  $1 \leq j \leq n_i$  (impulses) having the following properties. In the coordinates  $\mathbf{x}_i, \mathbf{y}_{ij}, \boldsymbol{\xi}_i, \boldsymbol{\eta}_{ij}$  on  $T^*Q \cong Q \times Q^*$ , the canonical symplectic structure  $\boldsymbol{\omega} = d\mathbf{p} \wedge d\mathbf{q}$ , the Hamiltonian function  $H$  of the  $N+1$  body problem and the first integral  $I$  of angular momentum (see (38) and (40)) have the form*

$$\boldsymbol{\omega} = \frac{\rho}{\omega} \tilde{\boldsymbol{\omega}}, \quad H = \frac{\rho}{\omega} \tilde{H}, \quad I = \frac{\rho}{\omega} \tilde{I} \quad (59)$$

where

$$\tilde{\boldsymbol{\omega}} = \boldsymbol{\omega}_0 + \varepsilon \boldsymbol{\omega}_1, \quad \tilde{H} = \omega \tilde{H}_0 + \varepsilon \tilde{H}_1 + \omega^2 \varepsilon \tilde{\Phi}, \quad \tilde{I} = I_0 + \varepsilon I_1, \quad (60)$$

$$\omega_0 = d\xi \wedge dx = \sum_{i=1}^n d\xi_i \wedge dx_i, \quad \omega_1 = d\eta \wedge dy = \sum_{i=1}^n \sum_{j=1}^{n_i} d\eta_{ij} \wedge dy_{ij},$$

$$\tilde{H}_0 = \sum_{i=1}^n \tilde{K}_i + \mu \sum_{1 \leq i < i' \leq n} K_{ii'}, \quad \tilde{H}_1 = \sum_{i=1}^n \left( \sum_{j=1}^{n_i} \tilde{S}_j^{(i)} + \nu \sum_{1 \leq j < j' \leq n_i} S_{jj'}^{(i)} \right), \quad (61)$$

$$I_0 = [\mathbf{x}, \xi] = \sum_{i=1}^n I_{i0}, \quad I_1 = [\mathbf{y}, \eta] = \sum_{i=1}^n \sum_{j=1}^{n_i} I_{ij}, \quad (62)$$

see (49), (50), (51). Here the small parameters  $0 < \omega, \varepsilon, \mu, \nu, \rho \ll 1$  are related by the conditions  $\rho = \omega^{2/3} \mu^{1/3}$  and  $\varepsilon = \omega^{1/3} \mu^{2/3} \nu = \nu \rho^2 / \omega$ ,  $\rho = \frac{1}{R}$ . The “perturbation potential”  $\tilde{\Phi} = \tilde{\Phi}(\mathbf{x}_*, \mathbf{y}_{**}, \bar{m}_*, \bar{m}_{**}, \mu, \nu, \rho)$  has the form (52), is an analytic function on the direct product of the regions

$$\left\{ \begin{array}{l} 1 + \nu \sum_{j'=1}^{n_i} \bar{m}_{ij'} \neq 0, \\ |\rho| \left( 1 + |\nu| \sum_{j'=1}^{n_i} |\bar{m}_{ij'}| \right) |\mathbf{y}_{ij}| < \min(|\mathbf{x}_i|, \frac{1}{2} \min_{i' \neq i} |\mathbf{x}_i - \mathbf{x}_{i'}|) \end{array} \right\}_{j=1}^{n_i},$$

$1 \leq i \leq n$ , and satisfies the condition

$$\tilde{\Phi}|_{\nu=\rho=0} = \sum_{i=1}^n \sum_{j=1}^{n_i} m_{ij} \left( F(\mathbf{x}_i, \mathbf{y}_{ij}) + \mu \sum_{\substack{i'=1 \\ i' \neq i}}^n \bar{m}_{i'} F(\mathbf{x}_i - \mathbf{x}_{i'}, \mathbf{y}_{ij}) \right), \quad (63)$$

see (43). In particular,  $\tilde{H}_1 = \tilde{\Phi} = 0$  in the case  $n_i = 0$  of a planetary system without satellites.

REMARK 3.2. Lemma 3.1 implies equivalences of the following  $(\varepsilon)$ -Hamiltonian systems for  $\omega, \varepsilon, \mu, \nu, \rho > 0$ :

$$(T^*Q, \omega, H) \cong (T^*Q, \tilde{\omega}, \tilde{H}) \cong (T^*Q, T^*Q_0, p; \omega_0, \omega_1; \tilde{H}, \tilde{H}_1 + \omega^2 \tilde{\Phi})^\varepsilon$$

where  $Q_0$  is the configuration space of planets, and  $p : T^*Q \rightarrow T^*Q_0$  is the projection. The third of these systems, called *the unperturbed system*, is not only Hamiltonian, but also  $\varepsilon$ -Hamiltonian (see (22)). Hence it naturally extends to any nonnegative values  $\omega, \varepsilon, \mu, \nu, \rho \geq 0$  of small parameters (despite of the fact that the symplectic structure degenerates if one of the parameters vanishes). For the limiting values of the parameters  $\omega > 0$  and  $\mu = \nu = 0$  (and, hence,  $\varepsilon = \rho = 0$ ), the third system becomes a 0-Hamiltonian system  $(T^*Q, T^*Q_0, p; \omega_0, \omega_1; \omega H_0, H_1 + \omega^2 \Phi)^0$  called *unperturbed* where  $H_0 := \tilde{H}_0|_{\mu=0}$ ,  $H_1 := \tilde{H}_1|_{\nu=0}$ ,  $\Phi := \tilde{\Phi}|_{\mu=\nu=\rho=0}$ . The system  $(T^*Q, T^*Q_0, p; \omega_0, \omega_1; \omega H_0, H_1)^0$ , called *the model system*, is  $\omega^2$ -close to the unperturbed system. It follows from lemma 3.1 that the unperturbed system indeed has the form (44) in the configuration space  $Q$ .

Due to (45) and (58), the functions  $\Phi_i, \Phi_{ii'}$  in (53), (54) have the following form for  $|\nu| \leq \nu_0, |\rho| \leq \rho_0$ :

$$\Phi_i = \nu \frac{m_i}{\bar{m}_i} F_{0,\nu\rho}(\mathbf{x}_i, \delta_i) + \sum_{j=1}^{n_i} \frac{m_{ij}}{\bar{m}_i} F_{0,\rho}(\mathbf{x}_i, \mathbf{y}_{ij} - \nu \delta_i) =$$

$$= \sum_{j=1}^{n_i} \frac{m_{ij}}{\bar{m}_i} F_{0,\rho}(\mathbf{x}_i, \mathbf{y}_{ij}) - \nu F(\mathbf{x}_i, \boldsymbol{\delta}_i) + O(\nu\rho), \quad (64)$$

$$\begin{aligned} \Phi_{ii'} = & \sum_{j=1}^{n_i} \frac{m_{ij}}{\bar{m}_i} F_{0,\rho}(\mathbf{x}_i - \mathbf{x}_{i'}, \mathbf{y}_{ij}) + \sum_{j'=1}^{n_{i'}} \frac{m_{i'j'}}{\bar{m}_{i'}} F_{0,\rho}(\mathbf{x}_{i'} - \mathbf{x}_i, \mathbf{y}_{i'j'}) - \\ & - \nu F(\mathbf{x}_i - \mathbf{x}_{i'}, \boldsymbol{\delta}_i) - \nu F(\mathbf{x}_{i'} - \mathbf{x}_i, \boldsymbol{\delta}_{i'}) + O(\nu\rho). \end{aligned}$$

Hence the “perturbation potential”  $\tilde{\Phi}$  in (52) satisfies the condition

$$\begin{aligned} \tilde{\Phi} = & \sum_{i=1}^n \sum_{j=1}^{n_i} m_{ij} \left( F_{0,\rho}(\mathbf{x}_i, \mathbf{y}_{ij}) + \mu \sum_{\substack{i'=1 \\ i' \neq i}}^n \bar{m}_{i'} F_{0,\rho}(\mathbf{x}_i - \mathbf{x}_{i'}, \mathbf{y}_{ij}) \right) - \\ & - \nu \sum_{i=1}^n \left( \bar{m}_i F(\mathbf{x}_i, \boldsymbol{\delta}_i) + \mu \sum_{\substack{i'=1 \\ i' \neq i}}^n \bar{m}_i F(\mathbf{x}_i - \mathbf{x}_{i'}, \boldsymbol{\delta}_i) \right) + O(\nu\rho), \quad \nu, \rho \rightarrow 0, \end{aligned} \quad (65)$$

which implies (63), see (46).

Let us explain a *geometric meaning* of the Hill potential  $F(\mathbf{x}, \mathbf{y})$  when the  $i$ th planet has  $n_i > 1$  satellites ( $1 \leq i \leq n$ ). Consider the function  $\Phi_i = \Phi_i(\mathbf{x}_i, \mathbf{y}_{i*}, \bar{m}_{i*}, \nu, \rho)$  defined by the formula (53) and called the “potential of interaction of all satellites of the  $i$ th planet with the Sun”. Due to (64), the function  $\Phi_i|_{\rho=0}$  equals a linear combination of the functions  $F(\mathbf{x}_i, \mathbf{y}_{ij})$ ,  $1 \leq j \leq n_i$ , and  $F(\mathbf{x}_i, \boldsymbol{\delta}_i)$ .

**3.3.1. The Poincaré transformation in the  $n+1$  body problem.** In order to prove lemma 3.1, we will explicitly construct the variables of impulses  $\boldsymbol{\xi}_i$ ,  $\boldsymbol{\eta}_{ij}$  and will show that the function  $H$  in (59), (60), (61) equals the total energy of the system. One easily shows that those summands in  $H$  that do not depend on the impulses give the potential energy  $U$ .

Let us compute the kinetic energy  $G$ . We will explore the fact that the transition from the coordinates  $(\mathbf{r}_0, \mathbf{r}_1, \dots, \mathbf{r}_N)$  in the configuration space to the coordinates  $\mathbf{x}_i, \mathbf{y}_{ij}$  (see §3.2.1) can be done by applying twice the following transformation called the Poincaré transformation.

Let us consider the configuration manifold  $Q$  of a planetary system (i.e. the system of  $n+1$  particles in a Euclidean plane). It consists of all ordered collections of radius vectors  $\mathbf{r}_0, \mathbf{r}_1, \dots, \mathbf{r}_n$  with associated masses  $c_0 = 1, c_1 = \lambda m_1, \dots, c_n = \lambda m_n$  where  $0 < \lambda \ll 1$ . Thus the manifold  $Q$  is naturally identified with the vector space  $\mathbb{R}^{2(n+1)}$  with coordinates  $\mathbf{r}_0, \mathbf{r}_1, \dots, \mathbf{r}_n$ . Consider the linear transformation  $L = L_{c_0, c_1, \dots, c_n}$  in  $\mathbb{R}^{2(n+1)}$  that corresponds to introducing the new linear coordinates on the space  $Q$  corresponding to the following collection of radius vectors:

$$\tilde{\mathbf{r}}_0 = \frac{\mathbf{r}_0 + c_1 \mathbf{r}_1 + \dots + c_n \mathbf{r}_n}{1 + c_1 + \dots + c_n}, \quad \tilde{\mathbf{r}}_1 = \mathbf{r}_1 - \mathbf{r}_0, \dots, \tilde{\mathbf{r}}_n = \mathbf{r}_n - \mathbf{r}_0$$

where  $\tilde{\mathbf{r}}_0 = \mathbf{c} := \frac{\mathbf{r}_0 + c_1 \mathbf{r}_1 + \dots + c_n \mathbf{r}_n}{1 + c_1 + \dots + c_n}$  is the radius vector of the centre of masses of the system.

DEFINITION 3.3. The transformation  $L = L_{c_0, c_1, \dots, c_n}$  is called *the Poincaré transformation* on the configuration manifold of the planetary system.

Actually one could consider another transformation, namely *the Jacobi transformation*  $\tilde{\mathbf{r}}_0 = \mathbf{c}$ ,  $\tilde{\mathbf{r}}_1 = \mathbf{r}_1 - \mathbf{c}$ ,  $\dots$ ,  $\tilde{\mathbf{r}}_n = \mathbf{r}_n - \mathbf{c}$ . But this transformation would lead to more awkward formulae. Moreover it would bring us to a desired result only in the case of a usual planetary system, i.e. having no satellites.

Consider the dual space  $Q^*$ , i.e. the space of all linear functions on the space  $Q$  (or, equivalently, the cotangent space to  $Q$  at its any point). This space consists of all collections  $\mathbf{p}_0, \mathbf{p}_1, \dots, \mathbf{p}_n$  whose each item  $\mathbf{p}_i$  is a linear function on the plane, i.e. a co-vector. In fact, we can define the value of the linear function corresponding to such a collection on the configuration  $(\mathbf{r}_0, \mathbf{r}_1, \dots, \mathbf{r}_n) \in Q$  to be  $\sum_{i=0}^n \langle \mathbf{p}_i, \mathbf{r}_i \rangle$ . It is clear that the nondegenerate transformation  $L$  on  $\mathbb{R}^{2(n+1)} \cong Q$  induces a linear transformation  $L^*$  on  $(\mathbb{R}^{2(n+1)})^* \cong Q^*$ . Denote the image of the collection  $\mathbf{p}_0, \mathbf{p}_1, \dots, \mathbf{p}_n$  under the transformation  $L^*$  by  $\tilde{\mathbf{p}}_0, \tilde{\mathbf{p}}_1, \dots, \tilde{\mathbf{p}}_n$ .

Consider the real valued function  $G = \sum_{i=0}^n \frac{\mathbf{p}_i^2}{2c_i}$  of kinetic energy on the space  $Q^*$ . Besides we consider the function  $I = \sum_{i=0}^n [\mathbf{r}_i, \mathbf{p}_i]$  of angular momentum on the space  $T^*Q$ . Finally consider the function of the total impulse  $\mathbf{P} = \mathbf{p}_0 + \mathbf{p}_1 + \dots + \mathbf{p}_n$  on  $Q^*$  whose values belong to the space of co-vectors, i.e. of linear functions on the plane.

LEMMA 3.2. Under the Poincaré transformation  $L = L_{c_0=1, c_1, \dots, c_n}$  on the configuration space  $Q$  of the  $n+1$  body problem, the functions  $G$  and  $\mathbf{P}$  on  $Q^*$  transform as follows:

(A) The kinetic energy  $G = \sum_{i=0}^n \frac{\mathbf{p}_i^2}{2c_i}$  has the form

$$G = \frac{\tilde{\mathbf{p}}_0^2}{2\tilde{c}_0} + \sum_{i=1}^n \frac{\tilde{\mathbf{p}}_i^2}{2\tilde{c}_i} + \frac{1}{2}(\tilde{\mathbf{p}}_1 + \dots + \tilde{\mathbf{p}}_n)^2 \quad (66)$$

where  $\tilde{c}_0 = 1 + c_1 + \dots + c_n = 1 + \lambda(m_1 + \dots + m_n)$ . The expression (66) can be rewritten as follows:

$$G = \frac{\tilde{\mathbf{p}}_0^2}{2\tilde{c}_0} + \sum_{i=0}^n \frac{\tilde{\mathbf{p}}_i^2}{2\tilde{c}_i} + \sum_{1 \leq i < i' \leq n} \langle \tilde{\mathbf{p}}_i, \tilde{\mathbf{p}}_{i'} \rangle \quad (67)$$

where  $\tilde{c}_i := c_0 c_i / (c_0 + c_i) = c_i / (1 + c_i)$ ,  $1 \leq i \leq n$ .

(B) The total impulse  $\mathbf{P} = \mathbf{p}_0 + \mathbf{p}_1 + \dots + \mathbf{p}_n$  transforms to the impulse of the “heaviest” particle:

$$\mathbf{P} = \tilde{\mathbf{p}}_0. \quad (68)$$

The function of angular momentum  $I = \sum_{i=0}^n [\mathbf{r}_i, \mathbf{p}_i]$  on the phase space  $X = T^*Q$  is  $L$ -invariant:  $I = \sum_{i=0}^n [\tilde{\mathbf{r}}_i, \tilde{\mathbf{p}}_i]$ .

PROOF. Items A and B directly follow by substituting into the functions  $G$  and  $\mathbf{P}$  the following explicit formulae for the transformation  $L^*$  of impulses:  $\mathbf{p}_0 = \frac{1}{1+c_1+\dots+c_n} \tilde{\mathbf{p}}_0 - \tilde{\mathbf{p}}_1 - \dots - \tilde{\mathbf{p}}_n$ ,  $\mathbf{p}_i = \tilde{\mathbf{p}}_i + \frac{c_i}{1+c_1+\dots+c_n} \tilde{\mathbf{p}}_0$ ,  $1 \leq i \leq n$ .

The invariance of the angular momentum follows from its invariance under the transformation on the space  $X = T^*Q$  induced by any linear transformation  $\tilde{\mathbf{r}}_i = \sum_j a_{ij} \mathbf{r}_j$  on the space  $Q$  with a nondegenerate matrix  $\|a_{ij}\|$ . The latter holds, since

the transformation of impulses has the form  $\tilde{\mathbf{p}}_i = \sum_k b_{ik} \mathbf{p}_k$  where  $\sum_i b_{ik} a_{ij} = \delta_{jk}$ , hence

$$\sum_{i=0}^n [\tilde{\mathbf{r}}_i, \tilde{\mathbf{p}}_i] = \sum_{i=0}^n \sum_{j=0}^n \sum_{k=0}^n [\mathbf{r}_j, \mathbf{p}_k] a_{ij} b_{ik} = \sum_{j=0}^n [\mathbf{r}_j, \mathbf{p}_j] = I. \quad \square$$

**PROOF OF LEMMA 3.1.** Let us consider the case of a planetary system without satellites. Observe that the transition from the coordinates  $\mathbf{r}$  to the coordinates  $\mathbf{x}$  is a composition of the Poincaré transformation  $L$  and the homothety  $\tilde{\mathbf{r}}_i = R\mathbf{x}_i$ ,  $1 \leq i \leq N$ . By setting  $\tilde{\mathbf{p}}_i = \sqrt{\frac{\mu}{R}} \boldsymbol{\xi}_i$ ,  $1 \leq i \leq N$ ,  $\tilde{\mathbf{p}}_0 = 0$ , one obtains from (67) the desired expression for the kinetic energy  $G$ . In fact,  $GR$  equals  $\sum_{i=1}^N \frac{\boldsymbol{\xi}_i^2}{2\bar{m}_i} + \mu \sum_{1 \leq i < j \leq N} \langle \boldsymbol{\xi}_i, \boldsymbol{\xi}_j \rangle$ . Hence, in the partial case of a planetary system without satellites, the function  $H$  in (59), (60), (61) indeed equals the total energy  $G + U$  of the system. The symplectic structure  $d\tilde{\mathbf{p}} \wedge d\tilde{\mathbf{r}} = \sqrt{\mu R} d\boldsymbol{\xi} \wedge d\mathbf{x}$  also has the desired form, since  $\sqrt{\mu R} = \frac{1}{\omega R}$ .

In the general case of a planetary system with satellites, we observe that a transition from the radius vectors  $(\mathbf{r}_0, \mathbf{r}_1, \dots, \mathbf{r}_N)$  to the coordinates  $\mathbf{x}_i, \mathbf{y}_{ij}$  can be obtained via performing the following three transformations. At first, one should perform the Poincaré transformation  $L$  to the whole system (see above). At second, one performs the transformation  $L$  to each satellite system  $\tilde{\mathbf{r}}_{ij}$ ,  $0 \leq j \leq n_i$ . Finally one performs the “scaling” homothety  $\tilde{\tilde{\mathbf{r}}}_{i0} = \mathbf{c}_i = R\mathbf{x}_i$ ,  $1 \leq i \leq n$ ,  $\tilde{\tilde{\mathbf{r}}}_{ij} = \mathbf{y}_{ij}$ ,  $1 \leq j \leq n_i$ . For the sake of simplicity, we will assume that all planets have satellites, i.e. all  $n_i$  are positive.

*Step 1.* The first performing of the Poincaré transformation  $L$  to the initial configuration variables gives, due to (66),

$$G = \frac{\tilde{\mathbf{p}}_0^2}{2\tilde{c}_0} + \sum_{i=1}^N \frac{\tilde{\mathbf{p}}_i^2}{2c_i} + \frac{1}{2}(\tilde{\mathbf{p}}_1 + \dots + \tilde{\mathbf{p}}_N)^2 = \frac{\tilde{\mathbf{p}}_0^2}{2\tilde{c}_0} + \sum_{i=1}^n \tilde{G}_i + \frac{1}{2}(\tilde{\mathbf{P}}_1 + \dots + \tilde{\mathbf{P}}_n)^2.$$

Here  $\tilde{G}_i = \frac{\tilde{\mathbf{p}}_i^2}{2c_i} + \sum_{j=1}^{n_i} \frac{\tilde{\mathbf{p}}_{ij}^2}{2c_{ij}} = \frac{\tilde{\mathbf{p}}_i^2}{2\mu m_i} + \sum_{j=1}^{n_i} \frac{\tilde{\mathbf{p}}_{ij}^2}{2\mu \nu \tilde{m}_{ij}}$  is the kinetic energy of the  $i$ th satellite system, and  $\tilde{\mathbf{P}}_i = \tilde{\mathbf{p}}_i + \sum_{j=1}^{n_i} \tilde{\mathbf{p}}_{ij}$  is its total impulse.

*Step 2.* Put  $\tilde{\mathbf{p}}_0 = 0$  and perform separately the Poincaré transformation  $L$  to each satellite system. As a result, we have from the formula (67)

$$\tilde{G}_i = \frac{\tilde{\mathbf{p}}_i^2}{2\mu \tilde{m}_i} + \sum_{j=1}^{n_i} \frac{\tilde{\mathbf{p}}_{ij}^2}{2\mu \nu \tilde{m}_{ij}} + \frac{1}{\mu m_i} \sum_{1 \leq j < j' \leq n_i} \langle \tilde{\mathbf{p}}_{ij}, \tilde{\mathbf{p}}_{ij'} \rangle$$

where  $\tilde{m}_i, \tilde{m}_{ij}$  are as in (48). By the formula (68), we have  $\tilde{\mathbf{P}}_i = \tilde{\tilde{\mathbf{p}}}_i$ .

By using the previous step, we obtain

$$\begin{aligned} G &= \sum_{i=1}^n \left( \frac{\tilde{\mathbf{p}}_i^2}{2\mu \tilde{m}_i} + \sum_{j=1}^{n_i} \frac{\tilde{\mathbf{p}}_{ij}^2}{2\mu \nu \tilde{m}_{ij}} + \sum_{1 \leq j < j' \leq n_i} \frac{\langle \tilde{\mathbf{p}}_{ij}, \tilde{\mathbf{p}}_{ij'} \rangle}{\mu m_i} \right) + \frac{1}{2} \left( \sum_{i=1}^n \tilde{\tilde{\mathbf{p}}}_i \right)^2 = \\ &= \sum_{i=1}^n \left( \frac{\tilde{\mathbf{p}}_i^2}{2\mu \tilde{m}_i} + \sum_{j=1}^{n_i} \frac{\tilde{\mathbf{p}}_{ij}^2}{2\mu \nu \tilde{m}_{ij}} + \sum_{1 \leq j < j' \leq n_i} \frac{\langle \tilde{\mathbf{p}}_{ij}, \tilde{\mathbf{p}}_{ij'} \rangle}{\mu m_i} \right) + \sum_{1 \leq i < i' \leq n} \langle \tilde{\tilde{\mathbf{p}}}_i, \tilde{\tilde{\mathbf{p}}}_{i'} \rangle \end{aligned}$$

where  $\tilde{m}_i = \frac{\bar{m}_i}{1+\mu\bar{m}_i}$ ,  $1 \leq i \leq n$ .

Step 3. Now perform the following “scaling” of coordinates and impulses:

$$\tilde{\mathbf{r}}_i = R\mathbf{x}_i, \quad \tilde{\mathbf{r}}_{ij} = \mathbf{y}_{ij}, \quad \tilde{\mathbf{p}}_i = \sqrt{\frac{\mu}{R}}\boldsymbol{\xi}_i, \quad \tilde{\mathbf{p}}_{ij} = \mu\nu\boldsymbol{\eta}_{ij} \quad (69)$$

for  $1 \leq i \leq n$ ,  $1 \leq j \leq n_i$ . This gives:

$$G = \sum_{i=1}^n \left( \frac{\boldsymbol{\xi}_i^2}{2\tilde{m}_i R} + \mu\nu \sum_{j=1}^{n_i} \frac{\boldsymbol{\eta}_{ij}^2}{2\tilde{m}_{ij}} + \frac{\mu\nu^2}{m_i} \sum_{1 \leq j < j' \leq n_i} \langle \boldsymbol{\eta}_{ij}, \boldsymbol{\eta}_{ij'} \rangle \right) + \frac{\mu}{R} \sum_{1 \leq i < i' \leq n} \langle \boldsymbol{\xi}_i, \boldsymbol{\xi}_{i'} \rangle.$$

By taking into account that  $\mu\nu\omega R = \varepsilon$ , we have the desired formula:

$$\omega RG = \sum_{i=1}^n \left( \omega \frac{\boldsymbol{\xi}_i^2}{2\tilde{m}_i} + \varepsilon \sum_{j=1}^{n_i} \frac{\boldsymbol{\eta}_{ij}^2}{2\tilde{m}_{ij}} + \frac{\varepsilon\nu}{m_i} \sum_{1 \leq j < j' \leq n_i} \langle \boldsymbol{\eta}_{ij}, \boldsymbol{\eta}_{ij'} \rangle \right) + \omega\mu \sum_{1 \leq i < i' \leq n} \langle \boldsymbol{\xi}_i, \boldsymbol{\xi}_{i'} \rangle.$$

Due to (69), the symplectic structure has the form

$$\omega = d\mathbf{p} \wedge d\mathbf{r} = d\tilde{\mathbf{p}} \wedge d\tilde{\mathbf{r}} = \sqrt{\mu R} \sum_{i=1}^n \left( d\boldsymbol{\xi}_i \wedge d\mathbf{x}_i + \varepsilon \sum_{j=1}^{n_i} d\boldsymbol{\eta}_{ij} \wedge d\mathbf{y}_{ij} \right),$$

since, recall,  $\varepsilon = \nu\sqrt{\frac{\mu}{R}}$ . This symplectic structure has the desired form (59), (60), (61), since  $\sqrt{\mu R} = \frac{1}{\omega R}$ . In a similar way, one proves the formulae for the first integral of angular momentum.

This finishes the proof of the main lemma 3.1.  $\square$

**3.3.2. Relations of the unperturbed system, the Hill problem, the three-body problem and the restricted three-body problem.** Recall that the unperturbed system (see (44) and remark 3.2) is a 0-Hamiltonian system consisting of  $n$  independent Kepler’s problems (for each planet) and  $N - n$  Hill’s problems (for each satellite of each planet). In particular, the Hill problem coincides with the unperturbed system for  $N = 2$  and  $n = 1$ , i.e. it can be obtained from the three-body problem (with gravitational constant  $g = 1/\mu$  and the relation (8) between small parameters) in a limit when  $\mu \rightarrow 0$  and  $\nu \rightarrow 0$ . As we will show below (see remark 3.3), the Hill problem can be obtained from the same 3-body problem in a limit when  $\mu \rightarrow 0$ .

Let us fix a real number  $\omega > 0$ , let the gravitational constant be equal to  $g = \frac{1}{\mu}$ , and study a limit of the Hamiltonian vector field in lemma 3.1 when  $\mu \rightarrow 0$  (or, equivalently, when the parameter  $\rho = \frac{1}{R} = \omega^{2/3}\mu^{1/3}$  approaches zero). Here  $\nu$  is arbitrary (and unnecessarily  $\nu \rightarrow 0$ ).

Denote  $\theta := \theta_1 = \nu m_{11}/(m_1 + \nu m_{11})$ ,  $m := \bar{m}_1 = m_1 + \nu m_{11}$ . Since  $N = 2$  and  $n = 1$ , the planet is a “double planet” ( $n_1 = 1$ ). Hence  $\Phi_1(\mathbf{x}, \mathbf{y}) = \frac{\theta(1-\theta)}{\nu} F_{\theta, \rho}(\mathbf{x}, \mathbf{y})$  and

$$\tilde{\Phi}(\mathbf{x}, \mathbf{y}) = \bar{m}_1 \Phi_1(\mathbf{x}, \mathbf{y}) = m \frac{\theta(1-\theta)}{\nu} F_{\theta, \rho}(\mathbf{x}, \mathbf{y}).$$

Here  $F_{\theta, \rho}(\mathbf{x}, \mathbf{y})$  is the generalized Hill potential (or the “potential of the interaction of the satellite and the Sun”), which is determined for  $0 < \theta < 1$  and  $\rho > 0$  by the relation

$$F_{\theta, \rho}(\mathbf{x}, \mathbf{y}) := \frac{1}{\theta(1-\theta)\rho^2} \left( \frac{1}{|\mathbf{x}|} - \frac{1-\theta}{|\mathbf{x} - \theta\rho\mathbf{y}|} - \frac{\theta}{|\mathbf{x} + (1-\theta)\rho\mathbf{y}|} \right), \quad (70)$$



see (45), (46). It follows from lemma 3.1 and the relations  $\varepsilon = \nu\rho^2/\omega$ ,  $\bar{m}_1 = m$ ,  $\tilde{m}_1 = \frac{m}{1+\mu m}$ ,  $\tilde{m}_{11} = \frac{m_{11}m_1}{m} = m\theta(1-\theta)/\nu$  that the 3-body problem under consideration with *positive* parameters  $0 < \theta < 1$ ,  $\mu > 0$ ,  $\omega > 0$  and  $\rho > 0$  related by the condition (8) is equivalent to the Hamiltonian system with the Hamiltonian function

$$\begin{aligned}\tilde{H} &= \omega \left( \frac{\xi^2}{2\tilde{m}_1} - \frac{\bar{m}_1}{|\mathbf{x}|} \right) + \varepsilon \left( \frac{\eta^2}{2\tilde{m}_{11}} - \frac{m_1 m_{11}}{|\mathbf{y}|} \right) + \omega^2 \varepsilon \tilde{\Phi}(\mathbf{x}, \mathbf{y}) = \\ &= \omega \left( (1 + \mu m) \frac{\xi^2}{2m} - \frac{m}{|\mathbf{x}|} \right) + \varepsilon \left( \frac{\eta^2}{2\tilde{m}_{11}} - \frac{m\tilde{m}_{11}}{|\mathbf{y}|} + \omega^2 \tilde{m}_{11} F_{\theta, \rho}(\mathbf{x}, \mathbf{y}) \right)\end{aligned}$$

and the symplectic structure  $\tilde{\omega} = d\xi \wedge d\mathbf{x} + \varepsilon d\eta \wedge d\mathbf{y}$ . Hence, it is described by the following system of equations:

$$\begin{cases} \frac{d^2 \mathbf{x}}{dt^2} = -\omega^2(1 + \mu m) \left( \frac{\mathbf{x}}{|\mathbf{x}|^3} + \theta(1 - \theta)\rho^2 \frac{\partial F_{\theta, \rho}}{\partial \mathbf{x}}(\mathbf{x}, \mathbf{y}) \right); \\ \frac{d^2 \mathbf{y}}{dt^2} = -m \frac{\mathbf{y}}{|\mathbf{y}|^3} - \omega^2 \frac{\partial F_{\theta, \rho}}{\partial \mathbf{y}}(\mathbf{x}, \mathbf{y}). \end{cases} \quad (71)$$

The power series expansion (45) of the function  $\frac{1}{|\mathbf{x} + \rho \mathbf{y}|}$  in the variable  $\rho$  at zero shows that the generalized Hill potential  $F_{\theta, \rho}(\mathbf{x}, \mathbf{y})$  extends on the region  $|\mathbf{x}| > |\rho \mathbf{y}|$ ,  $0 \leq \theta \leq 1$  to an analytic function in all arguments and satisfies (47). Therefore  $F_{\theta, 0} = F_{00} = F$ . This implies that the 3-body problem (71) continuously and analytically extends on the region of *nonnegative* values of parameters  $0 \leq \theta \leq 1$ ,  $\mu \geq 0$ ,  $\omega > 0$ ,  $\rho \geq 0$  related by the condition (8).

One obtains the following problems from the 3-body problem (71) for limiting parameter values: the *restricted three-body problem* for  $\theta = 0$ ,  $\mu > 0$ ,  $\omega > 0$ ,  $\rho > 0$ , (8), and the *Hill problem* for  $0 \leq \theta \leq 1$ ,  $\omega > 0$ ,  $\mu = \rho = 0$ . More precisely, in a limit when  $\omega > 0$ ,  $\mu \rightarrow 0$ , (8) (and, hence,  $\rho \rightarrow 0$ ), the three-body problem (71) uniformly in  $\theta \in (0, 1)$  tends to the following system of equations in the configuration space:

$$\begin{cases} \frac{d^2 \mathbf{x}}{dt^2} = -\omega^2 \frac{\mathbf{x}}{|\mathbf{x}|^3}; \\ \frac{d^2 \mathbf{y}}{dt^2} = -m \frac{\mathbf{y}}{|\mathbf{y}|^3} - \omega^2 \frac{\partial F}{\partial \mathbf{y}}(\mathbf{x}, \mathbf{y}), \end{cases} \quad (72)$$

i.e. to the Hill problem coinciding with (44) for  $N = 2$ ,  $n = n_1 = 1$ .

REMARK 3.3. (A) The Hill problem (72), in contrast to the three-body problem (with masses  $\mu_0 = 1$ ,  $\mu_1$ ,  $\mu_{11}$ ), does not depend on the parameter  $\theta = \frac{\mu_{11}}{\mu_1 + \mu_{11}}$  and, hence, on the ratio  $\frac{\mu_{11}}{\mu_1}$  of masses of the planet and the satellite. The Hill problem (72) formally depends on the parameters  $m > 0$  and  $\omega > 0$ . But, after the change  $\tilde{t} = \omega t$ ,  $\tilde{\mathbf{y}} = \frac{\omega^{2/3}}{m^{1/3}} \mathbf{y}$ ,  $\tilde{\mathbf{x}} = \mathbf{x}$ , it transforms into the similar problem with  $m = \omega = 1$ . The latter problem does not depend on any parameters and it is usually regarded as the Hill problem.

(B) The Hill problem was initially [1, 3] obtained as a limit when  $\mu \rightarrow 0$  and (8) from the *restricted three-body problem* (see (71) for  $\theta = 0$ ), but not from the three-body problem (71) itself. As we showed above, the three-body problem (71) tends to the Hill problem (72) when  $\mu \rightarrow 0$  and (8), provided that  $\theta \in [0, 1]$  is arbitrary (but not only  $\theta = 0$ ). This was observed already by Brown [18].

**3.4. Normalizing the Kepler problem near circular orbits.** It follows from lemma 3.1 and remark 3.2 that, for  $0 < \mu, \nu \ll \omega \ll 1$ , the  $N+1$  body problem of the type of planetary system with satellites is  $\omega^2$ -close to the corresponding *model system*  $(M_0 \times M_1, \omega_0 + \omega_1, \omega H_0 + H_1)$ , which is the direct product of independent Kepler's problems

$$\left( M_{i0}, d\xi_i \wedge d\mathbf{x}_i, \omega H_{i0} := \omega \tilde{K}_i \right), \quad \left( M_{ij}, d\boldsymbol{\eta}_{ij} \wedge d\mathbf{y}_{ij}, H_{ij} := \tilde{S}_j^{(i)} \right), \quad (73)$$

$1 \leq i \leq n$ ,  $1 \leq j \leq n_i$ , with the Hamiltonian functions (49). Let us show that the Hamiltonian function of any planar Kepler's problem has the form  $H_{ij} = H_{ij}(I_{ij}, q_{ij}, p_{ij})$  and satisfies the conditions (28) of theorem 2.1, with respect to some canonical coordinates  $\varphi_{ij}, I_{ij}, q_{ij}, p_{ij}$  on  $(M_{ij}, \boldsymbol{\omega}_{ij})$ .

The *planar Kepler problem* is given by the Hamiltonian system

$$\left( M = T^*(\mathbb{R}^2 \setminus \{0\}), \boldsymbol{\omega} = d\mathbf{p} \wedge d\mathbf{q}, H = \frac{\mathbf{p}^2}{2m} - \frac{km}{|\mathbf{q}|} = G + U \right). \quad (74)$$

Here  $G = \frac{\mathbf{p}^2}{2m}$  and  $U = -\frac{km}{|\mathbf{q}|}$  are the kinetic and potential energies of the system,  $\mathbf{q} \in \mathbb{R}^2 \setminus \{0\}$  and  $\mathbf{p} \in \mathbb{R}^2$  are the radius vector and the phase impulse of the particle,  $m, k > 0$  are parameters. The solutions of the Kepler problem having negative energy levels  $H$  are periodic. Since the Kepler problem is invariant under all rotations of the plane, it has the first integral of angular momentum  $I = [\mathbf{q}, \mathbf{p}]$ .

Note simple properties of circular motions in the Kepler problem:

1) For any  $r > 0$ , there is a unique (up to changing the direction of rotation) circular motion of the particle satisfying the system (74) along a circle of radius  $r$ . The angular velocity of this motion equals  $\Omega = \pm \sqrt{\frac{k}{r^3}}$ , while the energy  $H$  and the angular momentum  $I$  equal  $H = -\frac{km}{2r}$  and  $I = m\Omega r^2 = \pm m\sqrt{kr}$  respectively. In particular, the values  $H$  and  $I$  depend monotonically on the value  $\Omega$  (for  $\Omega > 0$  or  $\Omega < 0$ ) and take all values in the domains  $H < 0$  and  $I \neq 0$  respectively. We will assume that the parameters  $r, \Omega, H, I$  of a circular motion are related by the formulae from above.

2) Circular motions correspond to the equilibrium (i.e. stationary) positions of the particle with respect to a rotating coordinate system with angular velocity  $\Omega$ . Therefore, for any  $\alpha \neq 0$ , the solution of the Kepler problem corresponding to a circular motion is  $(T, \alpha)$ -periodic with  $T = \frac{\alpha}{\Omega}$ . In other words, such a solution is  $\frac{\alpha}{\Omega}$ -periodic with respect to a rotating coordinate system with angular velocity  $\Omega$  (as well as any angular velocity of the form  $(1 + \frac{2\pi\ell}{\alpha})\Omega$  where  $\ell$  is an integer).

The Kepler problem of our interest,  $(M_{ij}, d\boldsymbol{\eta}_{ij} \wedge d\mathbf{y}_{ij}, H_{ij})$  for  $1 \leq j \leq n_i$ , has the form (74) with  $M = M_{ij}, H = H_{ij}, I = I_{ij}$ ,

$$(\mathbf{q}, \mathbf{p}) = (\mathbf{y}_{ij}, \boldsymbol{\eta}_{ij}), \quad m = \tilde{m}_{ij}, \quad k = m_i + \nu m_{ij}, \quad r = r_{ij}, \quad \Omega = \Omega_{ij}. \quad (75)$$

(Similarly, the Kepler problem  $(M_{i0}, d\xi_i \wedge d\mathbf{x}_i, H_{i0})$  has the form (74) with  $M = M_{i0}, H = H_{i0}, I = I_{i0}, (\mathbf{q}, \mathbf{p}) = (\mathbf{x}_i, \xi_i), m = \tilde{m}_i, k = 1 + \mu \tilde{m}_i, r = R_i/R = \rho R_i = (1 + \mu \tilde{m}_i)^{1/3} |\Omega_{i0}|^{-2/3}$  and  $\Omega = \Omega_{i0} = \omega_i/\omega$ .)

**LEMMA 3.3 (NORMALIZING THE KEPLER PROBLEM).** *Let  $H$  and  $I$  be the Hamiltonian function and the first integral of angular momentum of the planar Kepler problem*

(74). In the region  $\{r > 0, p_\psi \neq 0\}$  of the phase space of the problem, consider the coordinates  $\varphi \bmod 2\pi$ ,  $I$ ,  $q$ ,  $p$  of the form

$$\varphi = \psi - \frac{2rp_r}{p_\psi}, \quad I = p_\psi, \quad q = \ln \frac{km^2 r}{p_\psi^2}, \quad p = rp_r. \quad (76)$$

Here  $r, \psi$  are the polar coordinates in the plane of motion,  $p_r, p_\psi$  are the corresponding impulses. Then:

(a) the coordinates (76) are canonical, i.e.  $\omega = dI \wedge d\varphi + dp \wedge dq$ ;

(b) in the coordinates (76), the Hamiltonian function  $H$  of the Kepler problem does not depend on the angular coordinate  $\varphi \bmod 2\pi$  and has the form

$$\frac{H}{k^2 m^3} = \frac{I^2 + p^2}{2I^4 e^{2q}} - \frac{1}{I^2 e^q} = -\frac{1}{2I^2} + \frac{q^2}{2I^2} + \frac{p^2}{2I^4} + o(q^2 + p^2)$$

as  $q^2 + p^2 \rightarrow 0$ . Furthermore the involutions  $S_l, S$  in §3.2.2 have the form

$$S_l(\varphi, I, q, p) = (-\varphi, -I, q, p), \quad S(\varphi, I, q, p) = (\varphi, -I, q, -p). \quad (77)$$

One proves lemma 3.3 in a direct way.  $\square$

The canonical coordinates  $\varphi \bmod 2\pi, I, q, p$  in lemma 3.3, as follows from their construction, are quite “similar” to the canonical coordinates  $\psi \bmod 2\pi, p_\psi = I, r, p_r$  corresponding to the polar coordinates  $\psi \bmod 2\pi, r$  in the configuration space of the planar Kepler problem. For example, the involutions  $S_l, S$  have the form (77) in the coordinates  $\psi, p_\psi, r, p_r$  too:

$$S_l(\psi, p_\psi, r, p_r) = (-\psi, -p_\psi, r, p_r), \quad S(\psi, p_\psi, r, p_r) = (\psi, -p_\psi, r, -p_r).$$

For any  $\Omega \neq 0$ , denote by  $\gamma_\Omega$  the phase trajectory of the Kepler problem corresponding to the circular motion with angular velocity  $\Omega$ . The invariant two-dimensional surface  $\cup_{\Omega \neq 0} \gamma_\Omega$  in the phase space formed by all these trajectories will be called *the surface of circular motions*. This surface is smooth and consists of two connected components, each of which is diffeomorphic to a punctured plane and bijectively projects onto the configuration manifold under the canonical projection.

We recall that the system describing the motion with respect to a rotating coordinate system with angular velocity  $\Omega$  is a Hamiltonian system with the Hamiltonian function  $H - \Omega I$ . Thus the circular motions correspond to the stationary points of such systems.

**COROLLARY 3.1.** *The surface of circular motions of the planar Kepler problem is a region in the symplectic coordinate cylinder:*

$$\bigcup_{\Omega \in \mathbb{R} \setminus \{0\}} \gamma_\Omega = \{(\varphi \bmod 2\pi, I, 0, 0) \mid I \neq 0\}.$$

For any  $\Omega \in \mathbb{R} \setminus \{0\}$ , the quadratic part (i.e. the quadratic form whose matrix is formed by the second partial derivatives) of the function  $H - \Omega I$  at any point of the circle  $\gamma_\Omega$  has diagonal form with respect to the coordinates (76):

$$\delta^2(H - \Omega I)|_{\gamma_\Omega} = -\frac{3}{mr^2} \delta I^2 + \Omega \left( I \delta q^2 + \frac{\delta p^2}{I} \right). \quad (78)$$

#### § 4. Deriving theorems 1.1–1.3 from theorems 2.1–2.4

Due to lemma 3.1 and remark 3.2, the  $N + 1$  body problem of the type of planetary system with satellites is equivalent to the  $\varepsilon$ -Hamiltonian system (33) with small parameters  $0 < \omega, \varepsilon, \mu, \nu, \rho \ll 1$  related by the conditions  $\varepsilon = \omega^{1/3} \mu^{2/3} \nu$  and  $\rho = \omega^{2/3} \mu^{1/3}$ . Moreover the functions  $\tilde{H}_0, \tilde{H}_1, \tilde{\Phi}$  are  $S^1$ -invariant, the function  $\tilde{H}_0 = H_0 + \mu R_0$  “projects” to  $M_0 := T^*Q_0$ , the function  $\tilde{H}_1 = H_1 + \nu R_1$  “projects” to  $M_1 := T^*Q_1$ , and their “principal parts” equal the sums  $H_0 = \sum_{i=1}^n H_{i0}$  and  $H_1 = \sum_{i=1}^n \sum_{j=1}^{n_i} H_{ij}$ . Furthermore, due to lemma 3.3, each summand has the form  $H_{ij} = \tilde{H}_{ij}(I_{ij}, q_{ij}, p_{ij})$  and satisfies the conditions (28). The perturbation potential  $\tilde{\Phi}$  is an analytic function in a neighbourhood of any torus  $\Lambda^\circ$ , provided that the collection of angular frequencies  $\Omega_{ij}$  satisfies the conditions (6) and (7) of “lack of collisions”. The principal part  $\Phi := \tilde{\Phi}|_{\mu=\nu=\rho=0}$  of the perturbation potential has the form (26).

So, the  $N + 1$  body problem of the type of planetary system with satellites considered in theorem 1.1 is equivalent to an  $\varepsilon$ -Hamiltonian system belonging to the class of “perturbed” systems in theorem 2.1.

PROOF OF THEOREMS 1.1 AND 1.2. *Step 1.* In theorems 1.1 and 2.1, the “relative resonance” conditions (9) and (29) on the collection of frequencies are equivalent. The nondegeneracy condition (11) from theorem 1.1 is equivalent to the nondegeneracy condition (30) from theorem 2.1.

Let us suppose that the more delicate nondegeneracy condition (12) from theorem 1.1 holds. Let us prove the nondegeneracy conditions (31) from theorem 2.1. The first condition in (31) is equivalent to the first condition in (12). In order to prove the second condition in (31), let us evaluate the number  $\Delta_{ij}$  in (32). Due to (63), we have  $F_{ij} = F_{ij}(\mathbf{x}_i, \mathbf{y}_{ij}) = m_{ij} F(\mathbf{x}_i, \mathbf{y}_{ij})$ . By construction,  $F_{ij}^\circ = (F_{ij}(\mathbf{x}_i, \cdot))|_{H_{ij}^{-1}(I_{ij}^\circ, 0, 0)}$  where  $\mathbf{x}_i = \text{const}$ ,  $|\mathbf{x}_i| = R_i/R = |\Omega_{i0}|^{-2/3}$  (see theorem 2.1). Let  $\langle F_{ij}^\circ \rangle = \langle F_{ij}^\circ \rangle(q_{ij}, p_{ij})$  be the function obtained by averaging the Hill potential  $F_{ij}^\circ = F_{ij}^\circ(q_{ij}, p_{ij})$  along the  $\frac{2\pi}{\Omega_{ij}}$ -periodic solutions of the Kepler problem  $(M_{ij}, \boldsymbol{\omega}_{ij}, H_{ij})$  for the satellite. By an easy calculation, taking into account (43) and corollary 3.1, we find the differential and the Hesse matrix of the function  $\langle F_{ij}^\circ \rangle$  at the point  $(0, 0)$ :

$$d\langle F_{ij}^\circ \rangle(0, 0) = 0, \quad \frac{\partial^2 \langle F_{ij}^\circ \rangle(0, 0)}{\partial(q_{ij}, p_{ij})^2} = \Omega_{i0}^2 \frac{I_{ij}}{\Omega_{ij}} \begin{pmatrix} -29/8 & 0 \\ 0 & 25/(8I_{ij}^2) \end{pmatrix}.$$

This and (78) imply that

$$\begin{aligned} \Delta_{ij} &= \frac{\Omega_{ij}}{2} \text{Tr} \left( \left( \frac{\partial^2 H_{ij}(I_{ij}^\circ, 0, 0)}{\partial(q_{ij}, p_{ij})^2} \right)^{-1} \frac{\partial^2 \langle F_{ij}^\circ \rangle(0, 0)}{\partial(q_{ij}, p_{ij})^2} \right) = \\ &= \frac{\Omega_{i0}^2}{2\Omega_{ij}} \text{Tr} \left( \begin{pmatrix} 1 & 0 \\ 0 & I_{ij}^2 \end{pmatrix} \begin{pmatrix} -29/8 & 0 \\ 0 & 25/(8I_{ij}^2) \end{pmatrix} \right) = -\frac{\Omega_{i0}^2}{4\Omega_{ij}}. \end{aligned}$$

Therefore the second desired condition in (31) has the form

$$\alpha + \Delta_{ij} \omega^2 T = \alpha - \frac{\omega_i^2}{4\Omega_{ij}} T \notin [-C_2 \omega^3 T, C_2 \omega^3 T] + 2\pi\mathbb{Z},$$

i.e. it is equivalent to the second condition in (12) with the constant  $C := C_2$ .

Thus, all the conditions of theorem 2.1 are fulfilled. Hence this theorem implies theorem 1.2(A).

*Step 2.* Let us check that, for the  $N + 1$  body problem under consideration, the model problem, the unperturbed and the perturbed problems are reversible. The construction of the functions  $\tilde{H}$ ,  $\tilde{H}_0$ ,  $\tilde{H}_1$  shows that they (and, hence, also  $\tilde{\Phi}$ ) are invariant under each of the involutions  $S_l$  and  $S$  from §3.2.2. Hence they are invariant under the composition  $J = S_l S = S S_l$ . Due to (77), the involution  $J = S_l S = S S_l$  acts component-wise in the form  $(\varphi_{ij}, I_{ij}, q_{ij}, p_{ij}) \mapsto (-\varphi_{ij}, I_{ij}, \tilde{q}_{ij}, -\tilde{p}_{ij})$ . From here, by taking into account the  $J$ -invariance of the functions  $\tilde{H}_0$ ,  $\tilde{H}_1$ ,  $\tilde{\Phi}$ , we obtain the reversibility of the model system, of the unperturbed and the perturbed systems. Hence theorem 2.2 implies theorem 1.1.

*Step 3.* Let us derive theorem 1.2(B) from theorem 2.3. By lemma 3.3 or (78), all numbers  $\frac{\partial^2 H_{ij}}{\partial I_{ij}^2}(I_{ij}^\circ, 0, 0)$  are negative. The sign  $\eta_{ij}$  from (34) equals

$$\eta_{ij} = \text{sgn} \left( \Omega_{ij} \text{Tr} \frac{\partial^2 H_{ij}(I_{ij}^\circ, 0, 0)}{\partial (q_{ij}, p_{ij})^2} \right) = \text{sgn} \Omega_{ij}.$$

Hence, by the first property of having fixed sign in theorem 1.2, all the signs  $\eta_{i0} = \text{sgn} \Omega_{i0}$  are the same, moreover  $\eta_{ij} \Delta_{ij} < 0$  for  $1 \leq j \leq n_i$ . Suppose that the conditions (15) and (16) hold for  $C := C_2$ . Then  $\alpha \notin \pi\mathbb{Z}$  and

$$\frac{\eta_{i0} + \eta_{ij}}{2} \alpha + \frac{\eta_{ij} \Delta_{ij}}{2} \omega^2 T \notin \left[ -\frac{C_2}{2} \omega^3 T, \frac{C_2}{2} \omega^3 T \right] + \pi\mathbb{Z}, \quad (79)$$

$$\frac{\eta_{ij} + \eta_{i'j'}}{2} \alpha + \frac{\eta_{ij} \Delta_{ij} + \eta_{i'j'} \Delta_{i'j'}}{2} \omega^2 T \notin [-C_2 \omega^3 T, C_2 \omega^3 T] + \pi\mathbb{Z} \quad (80)$$

for  $1 \leq j \leq n_i$  and  $1 \leq j' \leq n_{i'}$ . Consider any collection of real numbers  $\alpha_{ij}$ ,  $1 \leq i \leq n$ ,  $0 \leq j \leq n_i$ , such that

$$\alpha_{i0} = \eta_{i0} \alpha, \quad |\alpha_{ij} - \eta_{ij}(\alpha + \Delta_{ij} \omega^2 T)| \leq C_2 \omega^3 T, \quad 1 \leq i \leq n, \quad 1 \leq j \leq n_i.$$

Then:

- 1) the sum  $\alpha_{i0} + \alpha_{i'0} = 2\eta_{i0} \alpha$  does not belong to  $2\pi\mathbb{Z}$ , since  $\alpha \notin \pi\mathbb{Z}$ ;
- 2) for  $1 \leq j \leq n_i$ , the sum  $\alpha_{i'0} + \alpha_{ij} \in (\eta_{i0} + \eta_{ij})\alpha + \eta_{ij} \Delta_{ij} \omega^2 T + [-C_2 \omega^3 T, C_2 \omega^3 T]$  does not belong to  $2\pi\mathbb{Z}$  because of (79);
- 3) for  $1 \leq j \leq n_i$  and  $1 \leq j' \leq n_{i'}$ , the sum  $\alpha_{ij} + \alpha_{i'j'} \in (\eta_{ij} + \eta_{i'j'})\alpha + (\eta_{ij} \Delta_{ij} + \eta_{i'j'} \Delta_{i'j'}) \omega^2 T + [-2C_2 \omega^3 T, 2C_2 \omega^3 T]$  does not belong to  $2\pi\mathbb{Z}$  because of (80).

Thus, the sum of any two, possibly coinciding, numbers of the set  $\alpha_{ij}$  does not belong to the set  $2\pi\mathbb{Z}$ . Hence, the hypothesis of theorem 2.3 holds, and therefore this theorem implies theorem 1.2(B).  $\square$

**4.1. Necessity of the nondegeneracy condition  $\alpha \neq 0$ .** In this section, we give a more exact definition of the notion “almost any” in theorem 1.3 (meaning “any” for  $N \geq n = 2$ , see corollary 1.1(♯)) and of the subsets  $\mathcal{M} \subset \mathcal{M}^{\text{sym}} \subset \mathbb{R}_{>0}^n$ . Besides we derive theorem 1.3 and corollary 1.1(♯) from theorem 2.4.

Consider the sequence of *positive* real numbers

$$c_\kappa := \frac{\sqrt{r_\kappa}}{\pi} \int_{-\pi}^{\pi} \frac{r_\kappa + \frac{1}{r_\kappa} + \frac{r_\kappa + 3 \cos t}{2\kappa}}{(r_\kappa + \frac{1}{r_\kappa} - 2 \cos t)^{3/2}} \cos(\kappa t) dt \quad \text{where} \quad r_\kappa := \left( \frac{\kappa + 1}{\kappa} \right)^{2/3}, \quad (81)$$

$\kappa \in \mathbb{Z} \setminus \{-1, 0\}$ . It will interest us only up to a nonzero multiplicative factor. Here  $c_\kappa > 0$  due to the following properties of the integrand  $g_\kappa(t)$  in (81): it is an even function,  $g_\kappa(t) > 2|g_\kappa(2t + \frac{\pi}{2|\kappa|})|$  for  $0 < t < \frac{\pi}{2|\kappa|}$ , and  $g_\kappa(t) > |g_\kappa(t + \frac{\pi}{|\kappa|})|$  for  $\pi \frac{4\ell-1}{2|\kappa|} < t < \pi \frac{4\ell+1}{2|\kappa|}$ ,  $\ell = 1, 2, \dots, [\frac{|\kappa|}{2}]$ . The following property of the sequence  $c_\kappa$  is convenient to use for approximate computations:

$$c_\kappa = C_1 \kappa^2 + \frac{4}{3}(C_1 + C_2)\kappa + o(\kappa) \sim C_1 \kappa^2, \quad |\kappa| \rightarrow \infty,$$

where

$$C_1 := \frac{2}{\pi} \int_{-\infty}^{\infty} \frac{\cos t dt}{(\frac{4}{9} + t^2)^{3/2}} > 0, \quad C_2 := \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\cos t dt}{(\frac{4}{9} + t^2)^{5/2}} > 0.$$

On the basis of numeric computations, which were carried out by A. B. Kudryavtsev in the interval  $|\kappa| \leq 1000$ , one can make a conjecture that the sequence  $c_\kappa/\kappa^2$  decreases for  $0 < \kappa \rightarrow \infty$  and increases for  $-1 > \kappa \rightarrow -\infty$  to its limit  $C_1 \approx 2.18$ . (This is true in the indicated interval.) Approximate values of  $c_\kappa/\kappa^2$  for small  $|\kappa|$  are given in the following table:

$\kappa$	1	2	3	4	5	6	7	8	9	10
$c_\kappa/\kappa^2$	17.55	7.21	5.04	4.15	3.67	3.37	3.17	3.03	2.92	2.84
$\kappa$	-2	-3	-4	-5	-6	-7	-8	-9	-10	-11
$c_\kappa/\kappa^2$	0.37	0.72	0.97	1.16	1.29	1.39	1.48	1.54	1.60	1.64

Put

$$\kappa_{ii'} := \frac{\omega_i}{\omega_{i'} - \omega_i}, \quad i \neq i', \quad 1 \leq i, i' \leq n. \quad (82)$$

Then  $\frac{\omega_{i'}}{\omega_i} = \frac{\kappa_{ii'} + 1}{\kappa_{ii'}}$ ,  $\kappa_{i'i} = \kappa_{ii'} + 1$ , thus the numbers  $\kappa_{i'i}$  and  $\kappa_{ii'}$  are either both integer or both non-integer, moreover  $\kappa := \kappa_{ii'} \in \mathbb{R} \setminus \{0, -1\}$ . The number  $\frac{\kappa+1}{\kappa}$  equals the ratio  $\frac{\omega_{i'}}{\omega_i}$  of the angular frequencies of two planets along the circular orbits. Hence, due to Kepler's second law, the number  $r_\kappa = (\frac{\kappa+1}{\kappa})^{2/3}$  in (81) equals the ratio of the radii of these orbits. Let us define the number  $c_\kappa \in \mathbb{R}$  for any  $\kappa \in \mathbb{R}$  as follows: either by the formula (81) if  $\kappa \in \mathbb{Z} \setminus \{-1, 0\}$ , or by the formula

$$c_\kappa := 0 \quad \text{if} \quad \kappa \in \mathbb{R} \setminus (\mathbb{Z} \setminus \{-1, 0\}). \quad (83)$$

Consider the following collection of complex-valued functions on the  $n$ -dimensional torus  $(S^1)^n$  with angular coordinates  $\varphi = (\varphi_1, \dots, \varphi_n)$ :

$$f_{ll'}(\varphi) := \kappa_{ll'} c_{\kappa_{ll'}} e^{i\kappa_{ll'}(\varphi_{l'} - \varphi_l)}, \quad l \neq l', \quad 1 \leq l, l' \leq n, \quad (84)$$

where  $i = \sqrt{-1} \in \mathbb{C}$  is the imaginary unit and  $c_\kappa \geq 0$  is defined in (81) and (83).

The functions (84) are  $2\pi$ -periodic in each argument, moreover they are equivariant:

$$f_{ll'}(\varphi_1 + t, \dots, \varphi_n + t) = f_{ll'}(\varphi_1, \dots, \varphi_n), \quad t \in \mathbb{R},$$

$$f_{ll'}(\varphi_1 + \omega_1 t, \dots, \varphi_n + \omega_n t) = e^{i\omega_l t} f_{ll'}(\varphi_1, \dots, \varphi_n), \quad t \in \mathbb{R}.$$

DEFINITION 4.1. Let us fix the angular frequencies  $\omega_i$  of the planets satisfying the properties (9), (10), (6), (7). A collection of planets' masses  $\mu(m_1, \dots, m_n) \in \mathbb{R}_{>0}^n$  will be called *unclosing for a phase point*  $\varphi = (\varphi_1, \dots, \varphi_n) \in (S^1)^n$ , or simply  $\varphi$ -unclosing<sup>1</sup>, if at least one of the following numbers does not vanish:

$$f_l(\varphi; m_1, \dots, m_n) := \sum_{\substack{l'=1 \\ l' \neq l}}^n m_{l'} f_{ll'}(\varphi) \in \mathbb{C}, \quad 1 \leq l \leq n, \quad (85)$$

see (81)–(84). A phase point  $\varphi \in (S^1)^n$  will be called *symmetric* if it is fixed under the involution  $(S^1)^n \rightarrow (S^1)^n$ ,  $\varphi \mapsto -\varphi$ , of the  $n$ -dimensional torus, i.e. its coordinates have the form  $\varphi_l \in \{0, \pi\} \bmod 2\pi$ ,  $1 \leq l \leq n$ . Denote by  $\mathcal{M}$  (respectively  $\mathcal{M}^{\text{sym}}$ ) the set of all collections of planets' masses  $\mu(m_1, \dots, m_n) \in \mathbb{R}_{>0}^n$  that are  $\varphi$ -unclosing for any (respectively for any symmetric) phase point  $\varphi \in (S^1)^n$ .

REMARK 4.1. The number of symmetric phase points equals  $2^n$ . For any phase point  $\varphi \in (S^1)^n$ , the set of  $\varphi$ -closing collections of planets' masses is the intersection of a linear subspace of  $\mathbb{R}^n$  with  $\mathbb{R}_{>0}^n$ . Hence  $\mathcal{M}^{\text{sym}}$  is open in  $\mathbb{R}_{>0}^n$ . Moreover it is dense in  $\mathbb{R}_{>0}^n$  whenever it is nonempty. The subset  $\mathcal{M} \subset \mathbb{R}_{>0}^n$  is open in  $\mathbb{R}_{>0}^n$ , since the torus  $(S^1)^n$  is compact and the functions  $f_l = f_l(\varphi; m_1, \dots, m_n)$  are continuous, see (85). Suppose that  $\kappa_{ii'} \in \mathbb{Z}$  for some  $i \neq i'$ ,  $1 \leq i, i' \leq n$ . Then the set of  $\varphi$ -closing collections of planets' masses is contained in a plane of codimension  $\geq 2$  (since the system of functions (85) is linear in  $(m_1, \dots, m_n)$  and its rank is at least 2). Moreover any collection of masses with  $|\kappa_{ii'}|c_{\kappa_{ii'}}m_{i'} > \sum_{l \neq i, i'} |\kappa_{il}|c_{\kappa_{il}}m_l$  is  $\varphi$ -unclosing for any phase point  $\varphi \in (S^1)^n$  (i.e. belongs to  $\mathcal{M}$ ). Hence the open sets  $\mathcal{M} \subset \mathcal{M}^{\text{sym}}$  are nonempty, thus “almost any” collection of planets' masses (see the paragraph before theorem 1.3) belongs to  $\mathcal{M}^{\text{sym}}$ .

Consider the natural angular coordinates on the torus  $\Lambda^\circ$ :

$$x \mapsto \{\varphi_l, \varphi_{lj}, 1 \leq l \leq n, 1 \leq j \leq n_l\}, \quad x \in \Lambda^\circ.$$

The following statement generalizes theorem 1.3.

PROPOSITION 4.1. Consider the  $N + 1$  body problem of the type of planetary system with (or without) satellites,  $N \geq n \geq 2$ . Under the hypothesis of theorem 1.1, fix the angular frequencies  $\omega_1, \dots, \omega_n$  of planets having the form (9), (10), (6), (7). Suppose that there exists at least one pair of planets with indices  $i \neq i'$ , whose frequencies are in a special resonance (21). In this case, one automatically has  $\alpha = 0$ ,  $\kappa_{ii'} \in \mathbb{Z} \setminus \{0, -1\}$  and  $c_{\kappa_{ii'}} > 0$ . Fix the two-dimensional torus  $\gamma \subset \Lambda^\circ$  corresponding to a  $T$ -periodic solution of the model system. Let us suppose that the collection of planets' masses  $\mu(m_1, \dots, m_n) \in \mathbb{R}_{>0}^n$  is  $\varphi$ -unclosing for some (and, hence, any) point  $x = \{\varphi_l, \varphi_{lj}\} \in \gamma$ , see (85).

Then, for any real number  $D > 0$ , there exist numbers  $\mu_0, \nu_0 > 0$  and a neighbourhood  $U_0$  of the projection of the two-dimensional torus  $\gamma$  to the phase space of planets

<sup>1</sup>A collection of masses  $\mu(m_1, \dots, m_n) \in \mathbb{R}_{>0}^n$  is  $\varphi$ -closing if and only if, for any index  $l = 1, \dots, n$ , the planar polygonal line  $A_{l1}(\varphi) \dots A_{ln}(\varphi) \subset \mathbb{C}$  is closed, provided that the segments of this polygonal line have the form  $A_{ll'}(\varphi)A_{l, l'+1}(\varphi) = m_{l'} f_{ll'}(\varphi)$  for  $1 \leq l' \leq l-1$  and the form  $A_{l, l'-1}(\varphi)A_{ll'}(\varphi) = m_{l'} f_{ll'}(\varphi)$  for  $l+1 \leq l' \leq n$ .



such that the following property holds. For any values  $\mu, \nu$  such that  $0 < (\frac{\nu}{\nu_0})^3 \leq \mu \leq \mu_0$  (respectively  $0 < \mu \leq \mu_0$  if there are no satellites), the direct product  $U$  of the neighbourhood  $U_0$  and the phase space of satellites does not contain any  $(\tilde{T}, \tilde{\alpha})$ -periodic trajectory of the  $N + 1$  body problem under consideration, provided that the parameters  $\tilde{T}, \tilde{\alpha}$  have the form

$$|\tilde{T} - T| + |\tilde{\alpha}| \leq D\mu.$$

In particular, the assertions of theorem 1.3 hold.

Let us show that theorem 2.4 implies corollary 1.1( $\sharp$ ) about a planetary system with two planets without satellites ( $N = n = 2$ ). By lemma 3.1, the Hamiltonian function and the symplectic structure of the perturbed problem are

$$\omega \tilde{H}_0 = \omega \left( \tilde{K}_1 + \tilde{K}_2 + \mu K_{12} \right), \quad \omega_0 = d\xi_1 \wedge d\mathbf{x}_1 + d\xi_2 \wedge d\mathbf{x}_2.$$

Here  $\tilde{K}_i = \frac{\xi_i^2}{2\tilde{m}_i} - \frac{\tilde{m}_i}{|\mathbf{x}_i|}$  is the Hamiltonian function of the Kepler problem corresponding to the  $i$ th planet,  $\tilde{m}_i = \frac{\tilde{m}_i}{1+\mu\tilde{m}_i}$ ,  $i = 1, 2$ ,  $K_{12} = \langle \xi_1, \xi_2 \rangle - \frac{\tilde{m}_1\tilde{m}_2}{|\mathbf{x}_1 - \mathbf{x}_2|}$ .

Suppose that the ratio of frequencies  $\omega_1, \omega_2$  is rational, i.e. they have the form

$$\omega_1 = \frac{2\pi k_1}{T}, \quad \omega_2 = \frac{2\pi k_2}{T}$$

where  $k_1, k_2$  are nonzero integers,  $k_1 \neq \pm k_2$ ,  $T > 0$ . Then the solutions of the unperturbed problem ( $\mu = 0$ ) corresponding to the independent circular motions of planets with angular frequencies  $\omega_1, \omega_2$  are  $T$ -periodic. Conversely, if the solution is  $T$ -periodic then the pair of angular frequencies (5) is proportional to a pair of integers, with coefficient  $\frac{2\pi}{T}$ .

PROOF OF COROLLARY 1.1( $\sharp$ ). *Step 1.* Recall that the unperturbed problem (corresponding to  $\mu = 0$ ) splits into two independent planar Kepler's problem. Hence its  $T$ -periodic phase trajectories form the six-dimensional submanifold

$$\Theta = \{K_1 = \text{const}, K_2 = \text{const}\}$$

in the eight-dimensional phase space. In fact, due to periodicity of solutions of the Kepler problem with negative energy levels, the period of any its closed trajectory is a smooth (and strictly monotone, see §3.4) function in the value of energy.

Let us find the averaged perturbation  $\langle R_0^s \rangle$ , i.e. the function obtained by averaging the perturbation function  $R_0^s = R_0|_{\Theta} = (K_{12} + \frac{1}{2}\xi^2)|_{\Theta}$  along the periodic solutions of the unperturbed problem. In more detail, let us find the differential of the function  $\langle R_0^s \rangle$  at any point of the torus  $\Lambda^\circ \subset \Theta$ .

*Step 2.* With respect to polar coordinates  $\psi, r$  on the plane of motion, we have

$$\begin{aligned} \tilde{K}_i &= \frac{p_{r_i}^2 + p_{\psi_i}^2/r_i^2}{2\tilde{m}_i} - \frac{\tilde{m}_i}{r_i}, \quad i = 1, 2, \\ K_{12} &= \left( p_{r_1}p_{r_2} + \frac{p_{\psi_1}p_{\psi_2}}{r_1r_2} \right) \cos(\psi_1 - \psi_2) + \end{aligned}$$

$$+ \left( p_{r_1} \frac{p_{\psi_2}}{r_2} - \frac{p_{\psi_1}}{r_1} p_{r_2} \right) \sin(\psi_1 - \psi_2) - \frac{\bar{m}_1 \bar{m}_2}{\sqrt{r_1^2 + r_2^2 - 2r_1 r_2 \cos(\psi_1 - \psi_2)}}.$$

*Step 3.* Let us transfer to the coordinates  $\varphi_i, I_i, q_i, p_i, i = 1, 2$ , from lemma 3.3. By this lemma,

$$\Lambda^\circ = \{ p_1 = p_2 = q_1 = q_2 = 0, I_1 = \text{const}, I_2 = \text{const} \},$$

moreover the restriction of the linearized unperturbed system to the tangent bundle  $T_{\Lambda^\circ} \Theta = \{ \xi \mid dI_1(\xi) = dI_2(\xi) = 0 \}$  to  $\Theta$  has the following form (with respect to the coordinates  $\varphi_i, d\varphi_i, dq_i, dp_i, i = 1, 2$ , on this bundle):

$$\frac{d\varphi_i}{dt} = \omega_i, \quad \frac{d(d\varphi_i)}{dt} = 0, \quad \frac{d(dq_i)}{dt} = \omega_i \frac{dp_i}{I_i}, \quad \frac{1}{I_i} \frac{d(dp_i)}{dt} = -\omega_i dq_i, \quad (86)$$

$i = 1, 2$ . Hence, at each point  $(\varphi_1, \varphi_2)$  of the torus  $\Lambda^\circ$ , the differential of the function  $K_{12}|_\Theta$  has the following form:

$$\begin{aligned} d(K_{12}|_\Theta)(\varphi_1, \varphi_2) &= \frac{\bar{m}_1 \bar{m}_2 r_1 r_2}{r_{12}^3} \left( \sin \varphi_{12} \left( d\varphi_{12} + 2 \frac{dp_1}{I_1} - 2 \frac{dp_2}{I_2} \right) + \right. \\ &\quad \left. + \left( \frac{r_1}{r_2} - \cos \varphi_{12} \right) dq_1 + \left( \frac{r_2}{r_1} - \cos \varphi_{12} \right) dq_2 \right) - \\ &\quad - \frac{I_1 I_2}{r_1 r_2} \left( \cos \varphi_{12} (dq_1 + dq_2) + \sin \varphi_{12} \left( d\varphi_{12} + \frac{dp_1}{I_1} - \frac{dp_2}{I_2} \right) \right) \end{aligned} \quad (87)$$

where  $\varphi_{12} := \varphi_1 - \varphi_2$ ,  $r_{12} := \sqrt{r_1^2 + r_2^2 - 2r_1 r_2 \cos \varphi_{12}}$ . The perturbation function has the form  $R_0^\circ = R_0|_\Theta = (K_{12} + \frac{1}{2} \xi^2)|_\Theta$ . One easily shows that the contribution of the summand  $\frac{1}{2} \xi^2|_\Theta = \frac{1}{2} \sum_{i=1}^2 (p_{r_i}^2 + p_{\psi_i}^2 / r_i^2)|_\Theta$  to the averaged perturbation  $\langle R_0^\circ \rangle$  has a trivial differential at any point  $(\varphi_1, \varphi_2) \in \Lambda^\circ$ , i.e.  $d\langle R_0^\circ \rangle(\varphi_1, \varphi_2) = d\langle K_{12}|_\Theta \rangle(\varphi_1, \varphi_2)$ .

*Step 4.* Consider the rational number  $\kappa = \kappa_{12} = \frac{k_1}{k_2 - k_1} = \frac{\omega_1}{\omega_2 - \omega_1}$ , see (82). If  $\kappa \in \mathbb{Z}$  then define the number  $c_\kappa$  by the formula (81).

By integrating the values of the co-vector  $d(K_{12}|_\Theta)$  (see (87)) on the solutions of the linearized system (86), one obtains the following:

1) if  $\kappa \in \mathbb{Q} \setminus \mathbb{Z}$  then the differential of the function  $\langle R_0^\circ \rangle$  vanishes at any point of the two-dimensional torus  $\Lambda^\circ$ ;

2) if  $\kappa \in \mathbb{Z}$  then this differential has the following form at any point  $(\varphi_1, \varphi_2) \in \Lambda^\circ$ :

$$\begin{aligned} d\langle R_0^\circ \rangle(\varphi_1, \varphi_2) &= \bar{m}_1 \bar{m}_2 \left( \frac{\kappa c_\kappa}{r_1} \left( \cos(\kappa \varphi_{12}) dq_1 - \sin(\kappa \varphi_{12}) \frac{dp_1}{I_1} \right) - \right. \\ &\quad \left. - \frac{(\kappa + 1) c_{-\kappa-1}}{r_2} \left( \cos((\kappa + 1) \varphi_{12}) dq_2 - \sin((\kappa + 1) \varphi_{12}) \frac{dp_2}{I_2} \right) \right) \end{aligned} \quad (88)$$

and, hence, it does not vanish (since  $c_\kappa \neq 0$ ,  $c_{-\kappa-1} \neq 0$ , see (81)).

Since any point of the torus  $\Lambda^\circ$  is noncritical for the function  $\langle R_0^\circ \rangle$ , theorem 2.4 implies corollary 1.1( $\nabla$ ).  $\square$

PROOF OF PROPOSITION 4.1. By lemma 3.1, the summands  $\tilde{H}_0, \omega_0$  in the decompositions (60) of the Hamiltonian function and the symplectic structure of the  $N+1$  body problem have the form

$$\tilde{H}_0 = \sum_{i=1}^n \tilde{K}_i + \mu \sum_{1 \leq i < i' \leq n} K_{ii'}, \quad \omega_0 = \sum_{i=1}^n d\xi_i \wedge d\mathbf{x}_i.$$

Here  $\tilde{K}_i$  is the Hamiltonian function of the Kepler problem of the  $i$ th planet that is similar to the Hamiltonian function  $\tilde{K}_1$ , and  $K_{ii'}$  is the function similar to  $K_{12}$ . Put  $\kappa_{ii'} = \frac{\omega_i}{\omega_{i'} - \omega_i}$ ,  $1 \leq i, i' \leq n$ ,  $i \neq i'$ , see (82).

From the equality (88) in the case  $N = n = 2$ , we immediately obtain the analogous formula in general case  $N \geq n \geq 2$ :

$$d\langle R_0^\circ \rangle|_{\Lambda^\circ} = d\langle R_0^\circ \rangle(\varphi_1, \dots, \varphi_n) = \sum_{1 \leq i < i' \leq n} \xi_{ii'}^*(\varphi_i, \varphi_{i'}).$$

Here  $\xi_{ii'}^*(\varphi_i, \varphi_{i'})$  is the co-vector analogous to the co-vector (88), which is denoted by  $\xi_{12}^*(\varphi_1, \varphi_2)$ . In more detail, we have

$$d\langle R_0^\circ \rangle|_{\Lambda^\circ} = \sum_{i=1}^n \frac{\bar{m}_i}{r_i} \sum_{\substack{i'=1 \\ i' \neq i}}^n \bar{m}_{i'} \kappa_{ii'} c_{\kappa_{ii'}} \left( \cos(\kappa_{ii'} \varphi_{ii'}) dq_i - \sin(\kappa_{ii'} \varphi_{ii'}) \frac{dp_i}{I_i} \right)$$

where  $\varphi_{ii'} := \varphi_i - \varphi_{i'}$ . This co-vector vanishes at those points of the torus  $\Lambda^\circ$  where the functions  $f_i$ ,  $1 \leq i \leq n$ , simultaneously vanish, see (85). Hence theorem 2.4 implies the absence of  $(\tilde{T}, \tilde{\alpha})$ -periodic solutions in some neighbourhood of the torus  $\gamma$ , provided that the parameters  $\omega, \mu, \varepsilon > 0$  are small enough and are related by the inequalities  $\omega\varepsilon/\mu_0 \leq \mu \leq \mu_0$ . Due to the relation  $\varepsilon = \omega^{1/3} \mu^{2/3} \nu$ , these inequalities have the form  $\omega^{4/3} \mu^{2/3} \nu/\mu_0 \leq \mu \leq \mu_0$ , i.e. the form  $\omega^4 (\nu/\mu_0)^3 \leq \mu \leq \mu_0$ . Therefore theorem 2.4 indeed implies proposition 4.1.  $\square$

Theorem 1.3 obviously follows from proposition 4.1, definition 4.1 of the subsets  $\mathcal{M} \subset \mathcal{M}^{\text{sym}} \subset \mathbb{R}_{>0}^n$ , and remark 4.1.

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